POSET PINBALL, HIGHEST FORMS, AND $(n - 2, 2)$ SPRINGER VARIETIES

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ABSTRACT. In this manuscript we study type A nilpotent Hessenberg varieties equipped with a natural S^1-action using techniques introduced by Tymoczko, Harada-Tymoczko, and Bayegan-Harada, with a particular emphasis on a special class of nilpotent Springer varieties corresponding to the partition $\lambda = (n - 2, 2)$ for $n \geq 4$. First we define the **adjacent-pair matrix** corresponding to any filling of a Young diagram with n boxes with the alphabet $\{1, 2, \ldots, n\}$. Using the adjacent-pair matrix we make more explicit and also extend some statements concerning **highest forms of linear operators** in previous work of Tymoczko. Second, for a nilpotent operator N and Hessenberg function h, we construct an explicit bijection between the S^1-fixed points of the nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ and the set of (h, λ_N)-permissible fillings of the Young diagram λ_N. Third, we use **poset pinball**, the combinatorial game introduced by Harada and Tymoczko, to study the S^1-equivariant cohomology of type A Springer varieties $\mathcal{S}_{(n-2,2)}$ associated to Young diagrams of shape $(n - 2, 2)$ for $n \geq 4$. Specifically, we use the **dimension pair algorithm** for Betti-acceptable pinball described by Bayegan and Harada to specify a subset of the equivariant Schubert classes in the T-equivariant cohomology of the flag variety $\mathcal{F}lags(\mathbb{C}^n) \cong GL(n, \mathbb{C})/B$ which maps to a module basis of $H^*_{S^1}(\mathcal{S}_{(n-2,2)})$ under the projection map $H^*_T(\mathcal{F}lags(\mathbb{C}^n)) \to H^*_{S^1}(\mathcal{S}_{(n-2,2)})$. Our poset pinball module basis is not poset-upper-triangular; this is the first concrete such example in the literature. A straightforward consequence of our proof is that there exists a simple and explicit change of basis which transforms our poset pinball basis to a poset-upper-triangular module basis for $H^*_{S^1}(\mathcal{S}_{(n-2,2)})$. We close with open questions for future work.

CONTENTS

1. INTRODUCTION

The study of Hessenberg varieties is an active field of modern mathematical research. Indeed, Hessenberg varieties arise in many areas of mathematics, including geometric representation theory [8, 15, 16], numerical analysis [6], mathematical physics [12, 14], combinatorics [7], and algebraic geometry [4, 5], so it is of interest to explicitly analyze their topology, e.g. the structure of their (equivariant) cohomology rings. In this paper we further develop the approach, initiated and developed in [1, 2, 9, 10], which studies the topology of Hessenberg varieties through poset pinball and Schubert calculus techniques.

In this manuscript we focus on the case of nilpotent Hessenberg varieties, and more particularly on nilpotent Springer varieties. We begin by briefly recalling the setting of our results; for more details we refer the reader to Section 2. Let $N : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent operator. Let $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$

be a function satisfying $h(i) \geq i$ for all $1 \leq i \leq n$ and $h(i+1) \geq h(i)$ for all $1 \leq i < n$. In type A, **nilpotent Hessenberg varieties** can be defined as the following subvariety of $\mathcal{F}lags(\mathbb{C}^n)$:

$$\mathrm{Hess}(N, h) := \{V_\bullet = (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n) \mid NV_i \subseteq V_{h(i)} \text{ for all } i = 1, \ldots, n\}.$$

We equip $\mathrm{Hess}(N, h)$ with a natural S^1-action (described precisely in Section 2) induced from the diagonal torus subgroup T of $U(n, \mathbb{C})$ acting in the usual fashion on $GL(n, \mathbb{C})/B \cong \mathcal{F}lags(\mathbb{C}^n)$. In the special case when the Hessenberg function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the identity $h(i) = i$ for all $1 \leq i \leq n$, we call $\mathrm{Hess}(N, h)$ a **nilpotent Springer variety** and denote it by $\mathcal{S}_N$.

Our first two results apply to general type A nilpotent Hessenberg varieties. Let N be a nilpotent $n \times n$ matrix in Jordan canonical form with weakly decreasing block sizes and let λ denote the Young diagram[1] (equivalently the partition) with row lengths the Jordan block sizes of N listed in weakly decreasing order. In [17, Theorem 6.1] Tymoczko builds a paving-by-affines of a nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$, where the nilpotent operator N is required to be in **highest form** (see [17, Definition 4.1]). Much topological information about a variety is encoded in a paving-by-affines, so it is useful to build tools for dealing with the technical condition that the operator N be in highest form. Our first contribution is to introduce what we call the **adjacency-pair matrix**, which is an $n \times n$ matrix constructed from a filling of a Young diagram λ with n boxes by the alphabet $\{1, 2, \ldots, n\}$. This then allows us to make more explicit and also generalizes a procedure for producing highest forms of linear operators sketched in [17, Section 4]. In particular our methods allows us to straightforwardly derive the explicit change-of-basis permutation matrix which puts N into any choice of highest form (including that used by Tymoczko in [17], cf. Corollary 3.23). The adjacent-pair matrices also allows us to see precisely the set of permutation matrices which conjugate N to highest form (Theorem 3.21). The explicit nature of our results allows for other computations related to these nilpotent Hessenberg varieties. As an example, we derive in Lemma 3.28 an explicit formula for the Lie algebra projection induced by the inclusion of the S^1 subgroup acting on a special case of nilpotent Springer variety into the diagonal subgroup T of $U(n, \mathbb{C})$ acting on $\mathcal{F}lags(\mathbb{C}^n)$. Thus we expect our procedure to be useful for future poset pinball analysis of type A nilpotent Hessenberg varieties.

The affine cells in Tymoczko's paving-by-affines of $\mathrm{Hess}(N, h)$ are in one-to-one correspondence with **permissible fillings** of Young diagrams (defined precisely in Section 4); this is a useful combinatorial enumeration of the affine cells. The correspondence arises since the affine cells are intersections of $\mathrm{Hess}(N, h)$ with certain Schubert cells $BwB \subseteq GL(n, \mathbb{C})/B \cong \mathcal{F}lags(\mathbb{C}^n)$. Each such Schubert cell contains a unique (coset of a) permutation matrix wB, and each permutation w can be associated to a permissible filling of λ. Our second contribution is to extend this relationship between the permutations (which in this manuscript we think of as S^1-fixed points of $\mathrm{Hess}(N, h)$) and the permissible fillings as follows. For the purpose of the discussion below assume that N is in Jordan canonical form with weakly decreasing block sizes. We define for each permutation $\sigma \in S_n$ a bijection $\phi_{\lambda, \sigma}$ between the set $\mathcal{F}ill(\lambda)$ of fillings of λ with the set of permutations S_n (Definition 4.3). Each $\phi_{\lambda, \sigma}$ then induces a bijection between the permissible fillings $\mathcal{P}Fill(\lambda)$ of λ and the S^1-fixed points of the *translated* Hessenberg variety $\mathrm{Hess}(\sigma N \sigma^{-1}, h)$. In particular this yields an explicit formula for this bijection for all the possible highest forms of N in Theorem 3.21. Our results also provide proofs of statements quoted in [2].

Our third contribution is an explicit construction of a computationally convenient module basis for the S^1-equivariant cohomology[2] of a special class of type A nilpotent Springer varieties, namely, the 2-block (also known as 2-row) nilpotent Springer varieties associated to Young diagrams of the form $(n-2, 2)$, e.g.
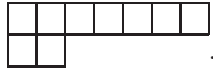
Here and below we will always assume $n \geq 4$, so the smallest Springer variety we consider corresponds to the 2×2 block

More specifically, we use the **poset pinball** methods introduced in [9] and the **dimension pair algorithm** for determining pinball rolldowns described in [2] to construct our combinatorially natural module basis for $H^*_{S^1}(\mathcal{S}_{(n-2,2)})$. Our arguments use our results above on highest forms and the explicit correspondence

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[1]We use English notation for Young diagrams.

[2]We work with cohomology with coefficients in $\mathbb{C}$ throughout, and hence omit it from our notation.

between permissible fillings and S^1-fixed points of the Springer variety. The module basis is obtained by taking images under the natural projection map $H^*_T(\mathcal{F}lags(\mathbb{C}^n)) \to H^*_{S^1}(\mathcal{S}_{(n-2,2)})$, to be described in detail below, of a subset of the T-equivariant Schubert classes in $H^*_T(\mathcal{F}lags(\mathbb{C}^n))$. A similar analysis by Bayegan and the second author in a special case of regular nilpotent Hessenberg varieties [2] yields a *poset-upper-triangular* basis in the sense of [9]. In contrast to the results in [2], in the present manuscript we find that the module basis is *not* poset-upper-triangular; this is the first such example in the literature. In addition, a straightforward consequence of our proof is that a simple change of variables yields a module basis which is not a poset pinball basis but is poset-upper-triangular. These results provides further evidence for the point of view, explained in [9], that geometrically natural GKM-type module bases in equivariant cohomology may not always be poset-upper-triangular, but still computationally convenient.

We now outline the contents of the paper. In Section 2 we provide the necessary definitions and set some notation. In Section 3 we define the adjacent-pair matrix and prove results concerning highest forms of linear operators. As a simple application we derive the change-of-variable matrix required to describe the circle subgroup of $T \subseteq U(n, \mathbb{C})$ acting on a translated Springer variety. Section 4 contains our results on the bijection between permissible fillings of a Young diagram and the S^1-fixed points of Hessenberg varieties. Section 5 is a mainly expository section which recalls the terminology and definitions of poset pinball and the dimension pair algorithm in [2,9]. In Sections 6 and 7, poset pinball for the case of $(n-2,2)$ Springer varieties is studied in detail. The small-n cases $n = 4$ and $n = 5$ are explicitly computed and recorded in Section 6. The main pinball result is in Section 7, where we prove that the dimension pair algorithm yields a linearly independent set of classes in $H^*_{S^1}(\mathcal{S}_{(n-2,2)})$ and hence a module basis. We close with some directions for future investigation in Section 8.

2. NILPOTENT HESSENBERG VARIETIES AND S^1-ACTIONS

We begin with the definition of the type A nilpotent Hessenberg varieties, of which the nilpotent Springer varieties are a special case. We also recall the definition of a circle subgroup of the maximal torus T of $U(n, \mathbb{C})$ which acts on any nilpotent Hessenberg variety. Since some of the discussion below applies to any nilpotent Hessenberg variety, we present the general definition here. We work exclusively with type A in this manuscript and hence omit it from our terminology below.

Given a nilpotent operator $N : \mathbb{C}^n \to \mathbb{C}^n$, consider its Jordan canonical form with weakly decreasing sizes of Jordan blocks. Let λ_N denote the partition of n with entries the sizes of the Jordan blocks of N. Throughout this manuscript we identify partitions of n with the corresponding Young diagram. For example, if $N : \mathbb{C}^5 \to \mathbb{C}^5$ is the operator with corresponding matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with respect to the standard basis of $\mathbb{C}^5$, then since the matrix has 2 Jordan blocks of sizes 3 and 2 respectively, it has associated Young diagram

which in turn corresponds to the partition $\lambda_N = (3, 2)$.

A **Hessenberg function** is a function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ satisfying $h(i) \geq i$ for all $1 \leq i \leq n$ and $h(i+1) \geq h(i)$ for all $1 \leq i < n$. We frequently denote a Hessenberg function by listing its values in sequence, $h = (h(1), h(2), \ldots, h(n) = n)$.

The **(nilpotent) Hessenberg variety** $\mathrm{Hess}(N, h)$ associated to N and a Hessenberg function h is a subvariety of the flag variety $\mathcal{F}lags(\mathbb{C}^n) \cong GL(n, \mathbb{C})/B$. Recall that $\mathcal{F}lags(\mathbb{C}^n)$ is the projective variety of nested

subspaces in $\mathbb{C}^n$, i.e.

$$\mathcal{F}lags(\mathbb{C}^n) = \{V_\bullet = (V_i) : 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n \text{ such that } \dim_{\mathbb{C}}(V_i) = i\}.$$

Then $\mathrm{Hess}(N, h)$ is defined to be the following subvariety of $\mathcal{F}lags(\mathbb{C}^n)$:

(2.1) $$\mathrm{Hess}(N, h) := \{V_\bullet \in \mathcal{F}lags(\mathbb{C}^n) \mid NV_i \subseteq V_{h(i)} \text{ for all } 1 \leq i \leq n\}.$$

The (nilpotent) Springer varieties[3] are Hessenberg varieties for the special case where the Hessenberg function is the identity function $h(i) = i$ for all $1 \leq i \leq n$:

Definition 2.1. Let $N : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent operator. The **Springer variety** $\mathcal{S}_N$ **associated to** N is defined as

$$\mathcal{S}_N := \{V_\bullet \in \mathcal{F}lags(\mathbb{C}^n) \mid NV_i \subseteq V_i \text{ for all } 1 \leq i \leq n\}.$$

For any $g \in GL(n, \mathbb{C})$, it is straightforward to see that the Hessenberg variety $\mathrm{Hess}(gNg^{-1}, h)$ for the conjugate gNg^{-1} of N is homeomorphic (in fact, isomorphic as algebraic varieties) to $\mathrm{Hess}(N, h)$, with explicit homeomorphism given by translation by g, i.e.,

(2.2) $$\mathrm{Hess}(N, h) \longrightarrow \mathrm{Hess}(gNg^{-1}, h)$$

$$h \longmapsto gh$$

where $h \in GL(n, \mathbb{C})$ denotes a flag $[h] \in GL(n, \mathbb{C})/B \cong \mathcal{F}lags(\mathbb{C}^n)$.

There exists a circle action on any nilpotent Hessenberg variety. Recall first that the maximal torus T of $U(n, \mathbb{C})$, identified with the diagonal subgroup of $U(n, \mathbb{C})$, acts on the flag variety $\mathcal{F}lags(\mathbb{C}^n)$. Consider the following circle subgroup of T:

(2.3) $$S^1 := \left\{ \begin{bmatrix} t^n & 0 & \cdots & 0 \\ 0 & t^{n-1} & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & t \end{bmatrix} \middle| \; t \in \mathbb{C}, \; \|t\| = 1 \right\} \subseteq T \subseteq U(n, \mathbb{C}).$$

It is shown in [9, Lemma 5.1] that the S^1 of (2.3) preserves the nilpotent Hessenberg variety $\mathrm{Hess}(N, h) \subseteq \mathcal{F}lags(\mathbb{C}^n)$ when the nilpotent operator N has matrix in Jordan canonical form with respect to the standard basis of $\mathbb{C}^n$. Moreover, the S^1-fixed points $\mathrm{Hess}(N, h)^{S^1}$ are isolated and are a subset of $\mathcal{F}lags(\mathbb{C}^n)^T$, the T-fixed points of $\mathcal{F}lags(\mathbb{C}^n)$. Using the identification $\mathcal{F}lags(\mathbb{C}^n)^T \cong S_n$ we henceforth think of S^1-fixed points of $\mathrm{Hess}(N, h)$ as permutations in S_n.

3. ADJACENT-PAIR MATRICES AND HIGHEST FORMS OF NILPOTENT OPERATORS

Suppose given a nilpotent matrix N_0 in standard Jordan canonical form with weakly decreasing Jordan block sizes. We think of N_0 as a linear operator on $\mathbb{C}^n$ written with respect to the standard basis of $\mathbb{C}^n$. As mentioned in Section 2, in addition to the Hessenberg variety $\mathrm{Hess}(N_0, h)$ we may also consider the translated Hessenberg varieties $\mathrm{Hess}(gN_0g^{-1}, h) = g \cdot \mathrm{Hess}(N_0, h)$ for various $g \in GL(n, \mathbb{C})$. For the purposes of poset pinball (discussed in more detail in Section 5) it turns out to be necessary to use conjugates $\sigma N \sigma^{-1}$ where σ is a permutation matrix and $\sigma N \sigma^{-1}$ is in so-called **highest form** [17, Definition 4.2]; this is because Tymoczko's construction of a paving-by-affines of a Hessenberg variety $\mathrm{Hess}(N, h)$ [17, Theorem 6.1] assumes that N is in highest form. Motivated by this, in this section we develop a theory which relates highest forms of N_0 with fillings of the corresponding Young diagram $\lambda = \lambda_{N_0}$. First we introduce a bijection $\phi_\lambda : \mathcal{F}ill(\lambda) \to S_n$ from the set of **fillings** $\mathcal{F}ill(\lambda)$ of λ and the permutation group S_n. Secondly we associate to each filling T of λ a matrix N_T which we call the **adjacent-pair matrix** of T. The main results of this section are Theorems 3.16 and 3.21. Theorem 3.16 observes that the adjacent-pair matrix N_T is precisely the conjugate $\sigma N_0 \sigma^{-1}$ where $\sigma = \phi_\lambda(T)$ is the permutation corresponding to T under the bijection ϕ_λ. This gives a computationally easy and explicit method for specifying the conjugates of N_0 by permutation matrices. In Theorem 3.21 we then prove that $N_T = \sigma N_0 \sigma^{-1}$ is in highest form precisely when T arises from a certain simple algorithm which we describe below. This yields a straightforward enumeration of all

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[3]In the literature they are also called *Springer fibres* because they arise as fibres of the symplectic resolution $T^*\mathcal{F}lags(\mathbb{C}^n) \to \mathcal{N}$ where $\mathcal{N}$ denotes the subspace of nilpotent matrices in $\mathfrak{gl}(n, \mathbb{C})$, but we do not need or use this perspective here.

permutation matrices σ for which $\sigma N_0 \sigma^{-1}$ is in highest form, and in particular in Corollary 3.25 we give a count of the number of conjugates $\sigma N_0 \sigma^{-1}$ for $\sigma \in S_n$ which are in highest form.

The discussion in this section has several motivations and consequences. Firstly, our results (e.g. Corollary 3.23) both make explicit and also generalize a procedure for producing highest forms of linear operators which is sketched in [17, Section 4, text near Figure 4]. Secondly, our explicit correspondence between certain fillings of λ and highest forms of N_0 allows us to easily determine the permutation $\sigma = \phi_\lambda(T)$ (see e.g. Example 3.26) and thus make further explicit computations with σ. As a sample such computation and for use in Section 7, at the end of this section we give a concrete description in coordinates of the conjugated circle $\sigma S^1 \sigma^{-1}$ which acts on the Springer variety $\mathcal{S}_{\sigma N_0 \sigma^{-1}}$ for N_0 corresponding to $\lambda = (n-2, 2)$, as well as a computation of the associated Lie algebra projection $\mathrm{Lie}(T) \to \mathrm{Lie}(\sigma S^1 \sigma^{-1})$. Thus some of the results in this section are preliminary to the arguments in the sections below. Third, we believe that the theory initiated here of highest forms in relation to Springer varieties is of independent interest; we describe some open questions motivated by this theory in Section 8.

We recall some definitions.

Definition 3.1. ([17, Definition 4.1]) Let X be any $m \times n$ matrix . We call the entry X_{ik} a **pivot** of X if X_{ik} is nonzero and if all entries below and to its left vanish, i.e., $X_{ij} = 0$ if $j < k$ and $X_{jk} = 0$ if $j > i$.

Moreover, given i, define r_i to be the row of $X_{r_i, i}$ if the entry is a pivot, and 0 otherwise.

Example 3.2. *Let*

$$X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Then $r_1 = 0$, $r_2 = 3$, $r_3 = 2$, and $r_4 = 4$.

Definition 3.3. ([17, Definition 4.2]) An upper-triangular nilpotent $n \times n$ matrix is in **highest form** if its pivots form a nondecreasing sequence, namely $r_1 \le r_2 \le \cdots \le r_n$.

Example 3.4. *The nilpotent matrix*

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in highest form since $r_1 = r_2 = r_3 = 0$, $r_4 = 2$, $r_5 = 3$, $r_6 = 5$.

Recall that a **filling** of λ by the alphabet $[n] := \{1, 2, \ldots, n\}$ is an injective placing of the integers $\{1, 2, \ldots, n\}$ into the boxes of λ. Following tableaux notation we denote by T a filling of λ by $[n]$. We denote by $\mathcal{F}ill(\lambda)$ the set of all fillings of λ by $[n]$. For λ a Young diagram with n boxes, we have $|\mathcal{F}ill(\lambda)| = n!$. In the theory below we use a particular bijective correspondence between $\mathcal{F}ill(\lambda)$ and S_n. We introduce the following terminology.

Definition 3.5. Let λ be a Young diagram. Let T be a filling of λ with alphabet $[n]$ for some $n \in \mathbb{N}$. By the **English reading** of T we mean the reading of the entries of T from left to right along rows, starting at the top row and proceeding in sequence to the bottom row. The word of T obtained via the English reading of T is called the **English word** of T. If λ is a Young diagram with n boxes then we define

(3.1) $$\phi_\lambda : \mathcal{F}ill(\lambda) \to S_n$$

where $\phi_\lambda(T)$ is the permutation whose one-line notation is given by the English word of T. Finally, if λ has n boxes then the **English filling** of λ is the filling T such that $\phi_\lambda(T)$ is the identity permutation in S_n.

For λ a Young diagram with n boxes, it is immediate from the definition that ϕ_λ is a bijection from $\mathcal{F}ill(\lambda)$ to S_n.

Example 3.6. *For*

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\\hline 5 & 6 \\\cline{1-2} 7 \\\cline{1-1}\end{array} \qquad \text{and} \qquad T' = \begin{array}{|c|c|c|c|}\hline 3 & 5 & 6 & 7 \\\hline 2 & 4 \\\cline{1-2} 1 \\\cline{1-1}\end{array}$$

we have that $\phi_\lambda(T)$ and $\phi_\lambda(T')$ are respectively the permutations (in one-line notation) 1234567 and 3567241. Moreover T is the English filling of $\lambda = (4, 2, 1)$.

Next we introduce a different reading of fillings which appears in the theory of highest forms and Hessenberg varieties developed by Tymoczko in [17] (but the terminology we use is new). In particular, this reading plays a significant role in our poset pinball methods in Sections 5-7 (cf. in particular Theorem 5.3).

Definition 3.7. Let λ be a Young diagram. Let T be a filling of a Young diagram with alphabet $[n]$ for some $n \in \mathbb{N}$. By the **rotated English reading** of T we mean the reading of the entries of T from the bottom to the top along columns, starting at the leftmost column and proceeding to the rightmost column. The word of T obtained via the rotated English reading is the **rotated English word** of T. Let λ be a Young diagram with n boxes. The **rotated English filling** of λ is the filling T of λ with $[n]$ such that its rotated English reading is the identity permutation in S_n.

Example 3.8. *Suppose that $\lambda = $. Then the rotated English filling of λ is the filling* $\begin{array}{|c|c|c|c|}\hline 3 & 5 & 6 & 7 \\\hline 2 & 4 \\\cline{1-2} 1 \\\cline{1-1}\end{array}$ *.*

Remark 3.9. *Note that the rotated English filling is not the same thing as the conjugate of the English filling of the conjugate Young diagram. For instance for the λ in Example 3.8 the conjugate of the English filling of the conjugate Young diagram $\tilde{\lambda}$ is* $\begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 7 \\\hline 2 & 5 \\\cline{1-2} 3 \\\cline{1-1}\end{array}$ *whereas the rotated English filling of λ is* $\begin{array}{|c|c|c|c|}\hline 3 & 5 & 6 & 7 \\\hline 2 & 4 \\\cline{1-2} 1 \\\cline{1-1}\end{array}$ *.*

Remark 3.10. *In the next section we develop a more general framework in which both Definition 3.5 and Definition 3.7 are special cases, but we do not need this perspective here.*

Given a Young diagram with n boxes and any filling T of λ by $[n]$, we now construct a matrix we call the **adjacent-pair matrix**. Our construction is a generalization of a procedure sketched by Tymoczko in [17, Section 4] (see in particular [17, Figure 4]). We begin by defining adjacency in λ and in a filling T.

Definition 3.11. Let λ be a Young diagram. We say that two boxes of λ are **adjacent** if the two boxes are in the same row, and one box is directly to the left of the other. That is, the two boxes are of the form $\begin{array}{|c|c|}\hline & \\\hline\end{array}$ within the Young diagram λ. Similarly, given a filling T of λ, we say that two entries of $\mathcal{M}$ are **adjacent**, or that they form an **adjacent pair**, if they occur in adjacent boxes.

Example 3.12. *For*

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\\hline 4 & 5 \\\cline{1-2} 6 \\\cline{1-1}\end{array}$$

the pairs $\{1, 2\}$, $\{2, 3\}$, and $\{4, 5\}$ are the adjacent pairs of entries of T.

Definition 3.13. Let λ be a Young diagram with n boxes and T a filling of λ with entries from $[n]$. Then we define the **adjacent-pair matrix** corresponding to T, denoted N_T, to be the matrix $N_T = (a_{ij})_{1 \le i,j \le n}$ such that its (i, j)-th entry is given by

$$a_{ij} := \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent in } T \text{ and } i \text{ is left of } j, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.14. *Suppose that* $\lambda =$ [diagram] *and that* $T =$ [tableau with 3,2,4 / 1,5 / 6] *. Then*

$$
N_T = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Remark 3.15. *The adjacent-pair matrix N_T corresponding to the English filling of λ is the nilpotent matrix in Jordan canonical form corresponding to λ. For example if* $T =$ [tableau with 1,2,3 / 4,5 / 6] *then*

$$
N_T = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The following is a basic computation which relates adjacent-pair matrices to highest forms. Given a permutation $\sigma \in S_n$ by slight abuse of notation we denote also by σ its $n \times n$ permutation matrix with respect to the standard basis of $\mathbb{C}^n$, i.e., the matrix with i-th column equal to the standard basis vector $e_{\sigma(i)}$.

Theorem 3.16. *Let N be an $n \times n$ nilpotent matrix in Jordan canonical form with weakly decreasing sizes of Jordan blocks. Let λ_N be the corresponding Young diagram and let $L_N : \mathbb{C}^n \to \mathbb{C}^n$ be the linear operator with matrix N with respect to the standard basis of $\mathbb{C}^n$. Let T be a filling on λ_N with alphabet $[n]$ and $\sigma := \phi_\lambda(T) \in S_n$ the permutation given by the English word of T. Then the adjacent-pair matrix N_T corresponding to T is equal to the conjugate $\sigma N \sigma^{-1}$, i.e., N_T is the matrix of L_N with respect to the basis $\{e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(n)}\}$.*

Proof. By the definition of the adjacent-pair matrix, given a Young diagram λ_N with n boxes and ℓ rows, N_T contains a 1 in $n - \ell$ entries, and all other entries are 0. Similarly, an $n \times n$ nilpotent matrix N, with corresponding Young diagram λ_N, contains a 1 in $n - \ell$ entries and 0's elsewhere, and so does any conjugate $\sigma N \sigma^{-1}$ for σ a permutation (matrix).

Now let a_{ij} denote the (i,j)-th entry of M. The preceding discussion implies that in order to prove the proposition it suffices to check that if $a_{ij} = 1$ for some i and j, then the matrix of L with respect to the basis $\{e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(n)}\}$ also contains a 1 at the (i,j)-th entry. Suppose that $a_{ij} = 1$. By construction this means that i and j are adjacent in the filling T with i to the left of j. Hence by definition of the English reading the one-line notation of σ is of the form $\sigma = \cdots i \, j \cdots$. Suppose that the i occurs at the ℓ-th spot of the one-line notation, so $\sigma(\ell) = i, \sigma(\ell+1) = j$. Then $\sigma^{-1}(i) = \ell, \sigma^{-1}(j) = \ell + 1$. Since i and j are adjacent, we also know that $L(e_{\ell+1}) = e_\ell$ (cf. Remark 3.15) or equivalently $L(e_{\sigma^{-1}(j)}) = e_{\sigma^{-1}(i)}$. This implies that the matrix of L written with respect to the basis $\{e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(n)}\}$ has a 1 in the (i,j)-th entry, as desired. □

We now wish to determine the set of fillings T such that the adjacent-pair matrix N_T is in highest form. Throughout this discussion we use the following assumptions and notation. Let λ be a Young diagram with n boxes, ℓ rows, k rows of distinct length, and r columns. If $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ are the distinct row lengths of λ we let d_i for $1 \leq i \leq k$ denote the number of rows of λ with length λ_i. Thus the row lengths of λ are

$$
(\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \lambda_2, \ldots, \lambda_2}_{d_2}, \cdots, \underbrace{\lambda_k, \lambda_k, \ldots, \lambda_k}_{d_k})
$$

with $\sum_{i=1}^k d_i = \ell$. We also let $(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$ denote the column lengths of λ. Note $\mu_1 = \ell$.

We begin with some observations about the pivots of an adjacent-pair matrix N_T.

Lemma 3.17. *Let λ be a Young diagram with n boxes and T a filling of λ by $[n]$. Let N_T be the adjacent-pair matrix of T. Then each matrix entry in N_T which is equal to 1 is a pivot of N_T.*

Proof. By definition of the adjacent-pair matrix, its non-zero entries are in one-to-one correspondence with the distinct adjacent pairs $\boxed{i \mid j}$ which appear in T. Each box of λ which is not in the leftmost (i.e. first) column of λ is the right hand box of precisely one such adjacent pair $\boxed{ \mid }$ of boxes. Hence each such accounts for precisely one entry of N_T equal to 1.

   By definition of fillings, each entry in T appears only once. In particular this means that for any $i \in [n]$, the index i appears at most once as either the right hand box $\boxed{j \mid i}$ or the left hand box $\boxed{i \mid j}$ in an adjacent pair in T. Thus by definition of the adjacent-pair matrix there exists at most one entry equal to 1 in each row and each column of N_T. Since all other entries are equal to 0, this in turn implies that each 1 that appears in N_T is in fact a pivot. □

Lemma 3.18. *Let λ be a Young diagram with n boxes and T a filling of λ by $[n]$. Let N_T be the adjacent-pair matrix of T. Then N_T is in highest form if and only if T satisfies the following conditions:*
   (a) *the leftmost column of λ is filled with the integers $\{1, 2, \ldots, \mu_1 = \ell\}$, and*
   (b) *if $\boxed{i_1 \mid j_1}$ and $\boxed{i_2 \mid j_2}$ both appear as adjacent pairs in T then*

$$i_1 < i_2 \text{ if and only if } j_1 < j_2.$$

Proof. First suppose N_T is in highest form. By Lemma 3.17 if $\boxed{i \mid j}$ appears as an adjacent pair in T then $r_j = i > 0$. For $j \in [n]$, the index j does not appear in the right hand box of any adjacent pair in T (so the j-th column of N_T is identically 0) precisely when j appears in the leftmost (i.e. first) column of λ. In this case, by definition of pivots, $r_j = 0$. Since N_T is in highest form we must have $r_1 \leq r_2 \leq \cdots \leq r_n$ and in particular any r_j with $r_j = 0$ must occur before any r_j with $r_j > 0$. We conclude that j is in the leftmost column of λ precisely when $1 \leq j \leq \mu_1 = \ell$. This proves (a). Now suppose $\boxed{i_1 \mid j_1}$ and $\boxed{i_2 \mid j_2}$ both appear as adjacent pairs in T. Then again from Lemma 3.17 we know $r_{j_1} = i_1, r_{j_2} = i_2$. If N_T is in highest form then the pivots must be increasing so $j_1 < j_2$ if and only if $i_1 < i_2$. This proves (b). If T satisfies conditions (a) and (b) then reversing this reasoning shows that N_T must be in highest form. □

   We now describe an algorithm which produces a filling T of λ which satisfies certain conditions, starting from the data of a filling of the leftmost column of λ. As we show in Theorem 3.21 below, the algorithm gives an explicit method for producing precisely those fillings T for which the corresponding N_T are in highest form. We follow notation established above.

(3.2)
   (1) Fix an arbitrary filling of the leftmost (i.e. first) column of λ with the alphabet $[\mu_1]$. This filling specifies a linear ordering of the rows of λ.
   (2) For the s-th column of λ for $2 \leq s \leq r$, place the μ_s integers $\{(\sum_{t=1}^{s-1} \mu_t) + 1, \ldots, \sum_{t=1}^{s} \mu_t\}$ in the μ_s boxes of the s-th column in the linear order specified by step (1).

Note that, by definition of this algorithm, the filling of the leftmost column completely specifies the rest of the filling.

Example 3.19. *If the Young diagram λ and the initial filling of its leftmost column are*

and
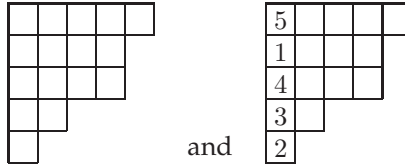
then the algorithm (3.2) *determines the rest of the filling to be*

.

Remark 3.20. *Suppose the filling of the leftmost column of λ is given by placing the integer i, for $1 \le i \le \mu_1$, in the i-th box from the bottom. Then the filling of λ obtained by applying the algorithm (3.2) is precisely the rotated English filling of Definition 3.7.*

We now prove that the fillings T for which N_T is in highest form are precisely those produced from the algorithm (3.2).

Theorem 3.21. *Let λ be a Young diagram with n boxes and T a filling of λ by $[n]$. Then the adjacent-pair matrix N_T is in highest form if and only if the algorithm (3.2) applied to the filling of the leftmost column of T produces the filling T.*

To prove the proposition we use the following lemma. We follow notation established above.

Lemma 3.22. *Let λ be a Young diagram with n boxes and T a filling of λ by $[n]$. Suppose T satisfies the conditions (a) and (b) of Lemma 3.18. Then the s-th column of λ for $1 \le s \le r$ contains precisely the integers $\{(\sum_{t=1}^{s-1} \mu_t) + 1, \ldots, \sum_{t=1}^{s} \mu_t\}$.*

Proof. We argue by induction. Condition (a) already implies the leftmost column is filled with $[\mu_1]$, which proves the base case $s = 1$. Now suppose the first s columns contain precisely the integers $\{1, 2, \ldots, \sum_{t=1}^{s} \mu_t\}$. Suppose for a contradiction that some element u in $\{(\sum_{t=1}^{s} \mu_t) + 1, \ldots, \sum_{t=1}^{s+1} \mu_t\}$ appears in the v-th column for some $v > s + 1$. Since there are precisely μ_{s+1} boxes in the $(s+1)$th column, this in turn implies that there must exist some $u' > \sum_{t=1}^{s+1} \mu_t$ that appears in the $(s+1)$th column. Thus there exist adjacent pairs $\boxed{i_1 \mid u'}$ and $\boxed{i_2 \mid u}$ with the properties that

- $i_1 \le \sum_{t=1}^{s} \mu_t$ and
- $i_2 > \sum_{t=1}^{s} \mu_t$

since u' appears in the $(s+1)$th column and all entries in the sth column are less than or equal to $\sum_{t=1}^{s} \mu_t$ by assumption, and since u appears in a column strictly to the right of the $(s+1)$th column. Thus $i_1 < i_2$ but $u' > u$, which contradicts condition (b). The result follows. $\qquad \square$

Proof of Theorem 3.21. By Lemma 3.18 it suffices to prove that a filling T satisfies conditions (a) and (b) of Lemma 3.18 if and only if it arises from (3.2). So suppose T satisfies Lemma 3.18(a) and (b). From Lemma 3.22 we already know that the set of entries in each column agrees with that specified by (3.2), so it remains to show that the ordering of the entries also agrees, i.e. that the entries of the s-th column for $2 \le s \le r$ respects the linear order imposed on the rows by the filling of the leftmost column. We argue by induction. Suppose $s = 2$. Then the entries of the 2nd column respect the ordering in the 1st column precisely when the following holds: if $\boxed{i_1 \mid j_1}$ and $\boxed{i_2 \mid j_2}$ are two adjacent pairs with j_1, j_2 in the 2nd column of λ then $i_1 < i_2$ if and only if $j_1 < j_2$. But this follows from condition (b). Moreover if this condition holds it follows that the linear ordering of the boxes in the 2nd column given by its filling by $\{\mu_1 + 1, \ldots, \mu_1 + \mu_2\}$ agrees with that induced by the linear ordering of the rows of λ corresponding to the filling of the 1st column. Assuming the first s columns are obtained by (3.2), the same argument as above shows that the $(s+1)$st column must also be filled according to (3.2), as desired.

Conversely, suppose T is obtained from (3.2). By construction T satisfies condition (a). Now suppose $\boxed{i_1 \mid j_1}$ and $\boxed{i_2 \mid j_2}$ are two adjacent pairs appearing in T. We consider cases. Suppose i_1 and i_2 appear in the sth and s'th columns of T. Without loss of generality we may assume $s < s'$. Then $i_1 \le \sum_{t=1}^{s} \mu_t$ while $i_2 \ge (\sum_{t=1}^{s'-1} \mu_t) + 1 \ge i_1$. Thus we wish to show $j_1 < j_2$. This follows because the adjacency with i_1 and i_2 respectively implies that j_1 is in the $(s+1)$th column and j_2 is in the $(s'+1)$th column. Since $s + 1 < s' + 1$ an argument similar to that above implies $j_1 < j_2$ as desired. On the other hand suppose i_1 and i_2 appear in the same column, say the sth. Then j_1 and j_2 appear in the $s + 1$th column. Suppose further that i_1 appears in the r_{i_1}th row and i_2 appears in the r_{i_2}th row. If $i_1 < i_2$ then by definition of the algorithm (3.2) the entry in the r_{i_1}th row of the first column is less than that in the r_{i_2}th row, which in turn implies $j_1 < j_2$. Similarly $j_1 < j_2$ implies $i_1 < i_2$. This concludes the proof. $\qquad \square$

The following, asserted in [17, Section 4, see e.g. Figure 4], is now a straightforward consequence.

Corollary 3.23. *Let λ be a Young diagram with n boxes and T_{RE} be the rotated English filling of λ. Let $\sigma := \phi_\lambda(T_{RE})$ be the permutation given by the English reading of T_{RE}. Then $N_{T_{RE}} = \sigma N \sigma^{-1}$ is in highest form.*

Proof. Immediate from Theorems 3.16 and 3.21 and Remark 3.20. □

We have just seen that each filling T obtained from (3.2) yields a conjugate $N_T = \sigma N \sigma^{-1}$ of N in highest form. Since a filling given in (3.2) is specified by the filling of its leftmost column, there are $\mu_1! = |S_{\mu_1}|$ many such fillings. However, different such fillings T and T' may yield the same adjacent-pair matrix $N_T = N_{T'}$. The next lemma makes this precise, for the purpose of which we use the following terminology. We say a filling T' is obtained from T by a **row swap** if the entries of 2 equal-length rows of T have been interchanged; more precisely, if both the ath row and the bth row of λ have d boxes and entries $\boxed{a_1\ \cdots\ a_d}$ and $\boxed{b_1\ \cdots\ b_d}$ respectively, then T' is obtained from T by swapping the ath row and bth row if T' contains the same entries as in T in all other rows, and the ath row of T' has entries $\boxed{b_1\ \cdots\ b_d}$ and the bth row has entries $\boxed{a_1\ \cdots\ a_d}$.

Lemma 3.24. *Let λ be a Young diagram with n boxes and let T and T' be fillings of λ obtained from (3.2). Then $N_T = N_{T'}$ if and only if T' is obtained from T by a sequence of row swaps.*

Proof. If T and T' differ only by a sequence of row swaps, then T and T' have precisely the same sets of adjacent pairs. Thus from the definition of the adjacent-pair matrix it follows that $N_T = N_{T'}$. Now suppose T and T' differ by more than a sequence of row swaps. Since both T and T' are obtained from (3.2), this means that there exists an element $s \in [\mu_1]$ which appears in T in a row of length d and appears in T' in a row of length d', with $d \neq d'$. Without loss of generality we assume $d' > d$. We wish to show that $N_T \neq N_{T'}$. For this it suffices to show that there exists some adjacent pair $\boxed{i\ j}$ which occurs in T but not in T', or vice versa. Consider the entries in the row of T and T' which contain s. By assumption these are of the form $\boxed{a_1 = s\ a_2\ \cdots\ a_d}$ and $\boxed{a'_1 = s\ a'_2\ \cdots\ a'_d\ \cdots\ a'_{d'}}$ respectively where $d' > d$. We take cases. Suppose there exists an index $1 < i \leq d$ for which $a_i \neq a'_i$. Then in particular there exists a minimal such, denote it i. Then there is an adjacent pair $\boxed{a_{i-1}\ a_i}$ in T and a pair $\boxed{a'_{i-1}\ a'_i}$ in T' where $a_{i-1} = a'_{i-1}$ but $a_i \neq a'_i$, so $N_T \neq N_{T'}$. Now suppose $a_i = a'_i$ for all $1 \leq i \leq d$. In particular $a_d = a'_d$. Then $\boxed{a'_d = a_d\ a'_{d+1}}$ is an adjacent pair in T' which does not occur in T. Hence $N_T \neq N_{T'}$ also in this case. The result follows. □

The following is now straightforward. Recall $\mu_1 = \ell$ is the total number of rows of λ and $d_1, \ldots, d_k$ are the numbers of rows of λ of length $\lambda_1, \ldots, \lambda_k$ respectively.

Corollary 3.25. *There exist precisely*

$$\frac{\ell!}{d_1! d_2! \cdots d_k!}$$

highest forms of N obtained as $\sigma N \sigma^{-1}$ for a permutation matrix $\sigma \in S_n$.

Proof. There are $\mu_1! = \ell!$ fillings T arising from the algorithm (3.2). From Lemma 3.24 we know that the matrices N_T do not change precisely when the entries in the first column contained in equal-length rows are permuted. The d_i count the numbers of equal-length rows so the result follows. □

Our constructions allow us to do explicit computations. For instance, given the discussion above it is straightforward to list the permutation matrices σ for which the associated conjugate $\sigma N \sigma^{-1}$ is in highest form. For instance, let T_{RE} be the rotated English filling of λ. It follows from the results above that the permutation σ for which $\sigma N \sigma^{-1}$ is the choice of highest form of N used in [17, Section 4] is precisely $\sigma := \phi_\lambda(T)$.

Example 3.26. *Suppose the Young diagram is*
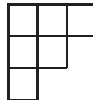
corresponding to the nilpotent matrix N in Example 3.4. Then the rotated English filling is

and the permutation σ such that $\sigma N \sigma^{-1}$ is in the highest form used in [17, Section 4] is $\sigma = 356241$.

As another application of our discussion and for use in Section 7 we close this section with a brief discussion about the circle action on Hessenberg varieties defined in (2.3). Consider the translated Hessenberg variety $\mathrm{Hess}(\sigma N \sigma^{-1}, h)$ where N is in standard Jordan canonical form and σ is a permutation matrix. In this case the circle subgroup of (2.3) does not necessarily act on $\mathrm{Hess}(\sigma N \sigma^{-1}, h)$. Instead we consider the conjugated circle subgroup $\sigma S^1 \sigma^{-1}$ of T, which is easily seen to preserve $\mathrm{Hess}(\sigma N \sigma^{-1}, h)$. Here and below we consider each such Springer variety to be equipped with this conjugated circle group action, which by slight abuse of notation we sometimes denote also by S^1 (instead of $\sigma S^1 \sigma^{-1}$). It is immediate that the fixed points $\mathrm{Hess}(\sigma N \sigma^{-1}, h)^{S^1}$ under the S^1-action are isolated and are a subset of $S_n \cong \mathcal{F}lags(\mathbb{C}^n)^T$; indeed, under the homeomorphism (2.2) the set of S^1-fixed points $\mathrm{Hess}(\sigma N \sigma^{-1}, h)^{S^1}$ is precisely the σ-translate

$$\sigma \cdot \mathrm{Hess}(N, h)^{S^1} \subseteq S_n$$

of the S^1-fixed points of $\mathrm{Hess}(N, h)$.

In Section 7 we focus attention on a choice of Springer variety $\mathcal{S}_{\sigma N \sigma^{-1}}$ specified by $\lambda = (n-2, 2)$ with nilpotent matrix N and the choice of permutation σ determined by the rotated English filling. In this setting we give below an explicit computation of the conjugate circle subgroup $\sigma S^1 \sigma^{-1}$ and also the associated linear projection $\mathrm{Lie}(T)^* \to \mathrm{Lie}(S^1)^*$. We illustrate with a concrete example.

Example 3.27. *Let* $\lambda = (4, 2)$. *Then the corresponding matrix in standard Jordan canonical form is*

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the associated permutation determined from the rotated English filling is $\sigma = 245613$. *The standard* S^1 *in* (2.3) *is then conjugated to the circle subgroup*

$$S^1 \cong \sigma S^1 \sigma^{-1} = \left\{ \begin{bmatrix} t^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^3 \end{bmatrix} \right\}.$$

The corresponding linear projection $\mathrm{Lie}(T^6)^* = \mathfrak{t}^* \to \mathrm{Lie}(S^1)^*$ *induced by the inclusion* $S^1 \cong \sigma S^1 \sigma^{-1} \hookrightarrow T^6$ *is given by*

(3.3) $$t_1 \mapsto 2t, \quad t_2 \mapsto 6t, \quad t_3 \mapsto t, \quad t_4 \mapsto 5t, \quad t_5 \mapsto 4t, \quad t_6 \mapsto 3t.$$

where t *denotes the variable in* $\mathrm{Lie}(S^1)$ *and the* t_i *the variables in* $\mathrm{Lie}(T^6) \cong \mathbb{R}^6$.

The general computation follows.

Lemma 3.28. *Let* $n \geq 4$. *Let* $\lambda = (n-2, 2)$ *and let* S^1 *denote the standard circle subgroup in* (2.3). *Then the permutation* σ *determined by the rotated English filling of* λ *is*

$$\sigma = 2\,4\,5\,6\,7\cdots n-1\,n\,1\,3$$

in one-line notation and the conjugated subgroup $\sigma S^1 \sigma^{-1}$ *is given by*

(3.4) $$S^1 \cong \sigma S^1 \sigma^{-1} = \left\{ \begin{bmatrix} t^2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ & t^n & & & & & \\ & & t & & & & \\ & & & t^{n-1} & & & \\ & & & & t^{n-2} & & \\ & & & & & \ddots & \\ & & & & & & t^3 \end{bmatrix} \right\} \subseteq T.$$

Moreover, the linear projection $\mathfrak{t}^* \to \mathrm{Lie}(S^1)^*$ *determined by the inclusion of this circle subgroup* $S^1 \cong \sigma S^1 \sigma^{-1} \hookrightarrow T$ *is given by*

$$(3.5) \qquad t_1 \mapsto 2t, \quad t_2 \mapsto nt, \quad t_3 \mapsto t, \quad \text{and } t_k \mapsto (n+3-k)t, \text{ for } 4 \le k \le n.$$

Proof of Lemma 3.28. By definition λ is the partition with $n-2$ boxes in the first row and 2 boxes in the second row. Its rotated English filling is
$\begin{array}{|c|c|c|c|c|c|c|} \hline 2 & 4 & 5 & \cdots & n-2 & n-1 & n \\ \hline 1 & 3 \\ \cline{1-2} \end{array}$
from which the form of σ (obtained by the English reading of the above filling) follows. Moreover the inverse of the given σ is $\sigma^{-1} = n - 1 \; 1 \; n \; 2 \; 3 \; 4 \cdots n-2$. The result follows by computation. $\square$

4. S^1-FIXED POINTS IN HESSENBERG VARIETIES AND PERMISSIBLE FILLINGS

In this section we give an explicit bijection from the S^1-fixed points of $\mathrm{Hess}(N, h)$, for various choices of N, to the set of **permissible fillings** of λ_N. The last result of the section, Corollary 4.10, is used in Sections 5-7 but the discussion is also of independent interest. Our results further develop some ideas in [17], in which Tymoczko constructs a paving-by-affines of a nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ by using certain Schubert cells. (In [17] Tymoczko considers more general Hessenberg varieties but we focus on the nilpotent case here.) Since each Schubert cell BwB in $GL(n, \mathbb{C})$ contains a unique coset wB with w a permutation matrix, it follows from her construction that there is a unique such w associated to each of the affine cells in her paving of $\mathrm{Hess}(N, h)$, which in turn can be encoded in a filling of a Young diagram [17, Theorem 7.1]. Our main result in this section, Theorem 4.7, is another interpretation of this bijection; our main contribution is to make more explicit and precise the bijective correspondence between the permissible fillings of λ_N and the cosets wB for w a permutation matrix which lie in $\mathrm{Hess}(\sigma N \sigma^{-1}, h)$ (thought of as S^1-fixed points of $\mathrm{Hess}(\sigma N \sigma^{-1}, h)$) for different choices of conjugates $\sigma N \sigma^{-1}$. We also refer the reader to [2] for related discussion; in particular, Corollary 4.10 proves a claim used in [2, Section 2].

We begin by defining permissible fillings following [13].

Definition 4.1. Let λ be a Young diagram with n boxes and $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ a Hessenberg function. A filling of λ is a (h, λ)-**permissible filling** if for every horizontal adjacency $\begin{array}{|c|c|}\hline k & j \\ \hline\end{array}$ we have $k \le h(j)$. (When the h and λ are understood from context we sometimes omit the (h, λ) from terminology and refer simply to **permissible fillings**.)

Remark 4.2. *In the context of Springer varieties, for which $h(j) = j$ for all j, the condition $k \le h(j)$ becomes $k \le j$. Thus in this case permissible fillings are precisely the **row-strict** fillings.*

Given λ and h, we denote by

$$\mathcal{P}Fill(\lambda, h) \subseteq \mathcal{F}ill(\lambda)$$

the set of permissible fillings of λ. Let N be a nilpotent $n \times n$ matrix in Jordan canonical form with corresponding Young diagram λ, and let $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a Hessenberg function. Our goal is to construct an explicit identification between Hessenberg fixed points $\mathrm{Hess}(\sigma N \sigma^{-1}, h)$ and permissible fillings $\mathcal{P}Fill(\lambda, h)$ for any permutation matrix σ.

As a first step we define an identification between S_n and $\mathcal{F}ill(\lambda)$ which depends on the choice of permutation σ. Recall that $\phi_\lambda : \mathcal{F}ill(\lambda) \to S_n$ is the mapping given by the English reading of a filling.

Definition 4.3. Let σ be a permutation in S_n and λ a Young diagram with n boxes. Consider the filling $\phi_\lambda^{-1}(\sigma)$ of λ corresponding to σ via the English reading. The filling $\phi_\lambda^{-1}(\sigma)$ specifies a linear ordering on the boxes of λ. Define the map

$$(4.1) \qquad \phi_{\lambda,\sigma} : \mathcal{F}ill(\lambda) \to S_n$$

by associating to any filling T of λ the permutation whose one-line notation is the reading of the entries of T with respect to the linear ordering given by $\phi_\lambda^{-1}(\sigma)$.

Example 4.4. *Suppose $\lambda = (3, 2, 1)$ and $\sigma = 253416$. Then $\phi_\lambda^{-1}(\sigma)$ is the filling* $\begin{array}{|c|c|c|}\hline 2 & 5 & 3 \\ \hline 4 & 1 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$ *so for the filling $T =$*

$\begin{array}{|c|c|c|}\hline 4 & 1 & 6 \\ \hline 2 & 3 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$ *the reading $\phi_{\lambda,\sigma}(T)$ would yield* 346215.

Remark 4.5. *By definition the mapping $\phi_{\lambda,id}$ corresponding to $\sigma = id$ the identity permutation coincides with the map ϕ_λ obtained via the English reading. Similarly the permutation σ for which $\phi_{\lambda,\sigma}(T)$ is the rotated English reading is precisely the permutation corresponding under ϕ_λ to the rotated English filling of λ.*

Remark 4.5 shows that both the English and the rotated English readings of $\mathcal{F}ill(\lambda)$ are special cases of $\phi_{\lambda,\sigma}$. The point of Definition 4.3 is to emphasize that other choices, corresponding to different choices of translated Hessenberg varieties, are possible. We need the following lemma.

Lemma 4.6. *Let λ be a Young diagram with n boxes and $\sigma, \tau \in S_n$. Then*

$$\phi_{\lambda,\sigma}^{-1}(\tau) = \phi_\lambda^{-1}(\tau\sigma).$$

Proof. This follows from the definition of $\phi_{\lambda,\sigma}$ and the fact that multiplication by σ on the right re-orders the entries in the one-line notation for τ precisely by replacing the i-th entry $\tau(i)$ by $\tau(\sigma(i))$ for all i. $\qquad\square$

The main theorem of this section is the following. We consider $\text{Hess}(\sigma N \sigma^{-1}, h)^{S^1}$ to be a subset of S_n and $\mathcal{P}Fill(\lambda)$ to be a subset of $\mathcal{F}ill(\lambda)$.

Theorem 4.7. *Let N be an $n \times n$ nilpotent matrix in Jordan canonical form with weakly decreasing sizes of Jordan blocks with respect to the standard basis of $\mathbb{C}^n$ and let $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a Hessenberg function. Let $\mathcal{P}Fill(\lambda, h)$ denote the corresponding set of permissible fillings of λ. Let $\sigma \in S_n$ and denote by $\text{Hess}(\sigma N \sigma^{-1}, h)$ the associated the nilpotent Hessenberg variety equipped with the S^1-action described in Section 2. Then the assocation*

$$(4.2) \qquad\qquad \Phi_{\lambda,\sigma} : w \mapsto \phi_{\lambda,\sigma}^{-1}(w^{-1})$$

defines a bijection from $\text{Hess}(\sigma N \sigma^{-1}, h)^{S^1}$ to $\mathcal{P}Fill(\lambda, h)$.

In the proof of Theorem 4.7 we use the following terminology. Suppose $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a Hessenberg function. We define the **Hessenberg space** H corresponding to h to be the subspace of $\mathfrak{gl}(n, \mathbb{C})$ defined by

$$(4.3) \qquad\qquad H := \{X \in \mathfrak{gl}(n, \mathbb{C}) \,|\, X_{ij} = 0 \text{ if } i > h(j)\}$$

where X_{ij} denotes the (i, j)-th entry of the matrix X.

Example 4.8. *Suppose $h = (2, 3, 4, 4)$. Then*

$$H = \{X \in \mathfrak{gl}(4, \mathbb{C}) \,|\, X_{3,1} = X_{4,1} = X_{4,2} = 0\} = \left\{ \begin{bmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \end{bmatrix} \right\} \subseteq \mathfrak{gl}(n, \mathbb{C})$$

where the $\star$ denotes free variables.

It is straightforward to reformulate the definition (2.1) of Hessenberg varieties as follows: for a given Hessenberg function h with corresponding Hessenberg space H,

$$(4.4) \qquad\qquad \text{Hess}(N, h) = \{[g] \in GL(n, \mathbb{C})/B \,|\, g^{-1}Ng \in H\}.$$

In particular, the S^1-fixed points of $\text{Hess}(N, h)$ are precisely

$$(4.5) \qquad\qquad \text{Hess}(N, h)^{S^1} \cong \{w \in S_n \,|\, w^{-1}Nw \in H\}.$$

We use the following lemma.

Lemma 4.9. *Let λ be a Young diagram with n boxes and $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ a Hessenberg function with corresponding Hessenberg space H. Let T be a filling of λ by the alphabet $[n]$ and let M be the $n \times n$ matrix obtained by applying the adjacency algorithm to T. Then*

$$T \text{ is } (h, \lambda)\text{-permissible} \quad \Longleftrightarrow \quad M \in H.$$

Proof. By definition of the adjacency algorithm, the (i, j)-th entry of M is non-zero precisely when $\boxed{i \,|\, j}$ occurs in the filling of T. Hence by definition of H the matrix M is in H precisely if, for all such adjacent pairs (i, j) in T, we have $i \leq h(j)$. This is exactly the definition of a (h, λ)-permissible filling. $\qquad\square$

Proof of Theorem 4.7. We first prove the claim for the special case $\sigma = id$. In this case $\phi_{\lambda,id} = \phi_\lambda$ (cf. Remark 4.5) and we wish to show that the association $w \mapsto \Phi_{\lambda,id}(w) := \phi_\lambda^{-1}(w^{-1})$ defines a bijection between $\mathrm{Hess}(N,h)^{S^1}$ and $\mathcal{P}Fill(\lambda,h)$. Since taking inverses is a bijection on S_n and ϕ_λ is also a bijection from $Fill(\lambda)$ to S_n, the content of the claim is that a permutation w is in $\mathrm{Hess}(N,h)^{S^1}$ precisely when the filling $\phi_\lambda^{-1}(w^{-1})$ is permissible. Recall from (4.5) that

$$w \in \mathrm{Hess}(N,h)^{S^1} \quad \Longleftrightarrow \quad w^{-1}Nw \in H.$$

By Theorem 3.16, the matrix $w^{-1}Nw$ is precisely the adjacent-pair matrix for the filling $\phi_\lambda^{-1}(w^{-1})$. The claim now follows from Lemma 4.9.

The claim for $\Phi_{\lambda,\sigma}$ for arbitrary $\sigma \in S_n$ follows from the special case $\Phi_{\lambda,id}$ because

$$\mathrm{Hess}(\sigma N \sigma^{-1}, h)^{S^1} = \sigma \cdot \mathrm{Hess}(N,h)^{S^1}$$

and

$$\phi_{\lambda,\sigma}^{-1}((\sigma \cdot w)^{-1}) = \phi_{\lambda,\sigma}^{-1}(w^{-1}\sigma^{-1}) = \phi_\lambda^{-1}(w^{-1})$$

where the last equality uses Lemma 4.6. This completes the proof. $\square$

The following is used below in Sections 5-7 as well as in [2]. Given a Young diagram λ with n boxes, denote by T_{RE} the rotated English filling of λ.

Corollary 4.10. *Let N be an $n \times n$ nilpotent matrix in Jordan canonical form and weakly decreasing sizes of Jordan blocks with respect to the standard basis of $\mathbb{C}^n$ and let $h : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ be a Hessenberg function. Let $\mathcal{P}Fill(\lambda,h)$ denote the corresponding set of permissible fillings of λ. Let $\sigma = \phi_\lambda(T_{RE})$ be the permutation corresponding to the rotated English filling of λ. Then*

(4.6) $$\Phi_{\lambda,\sigma} : w \mapsto \phi_{\lambda,\sigma}^{-1}(w^{-1})$$

is a bijection from $\mathrm{Hess}(\sigma N \sigma^{-1}, h)^{S^1}$ to $\mathcal{P}Fill(\lambda,h)$.

5. BETTI-ACCEPTABLE PINBALL AND LINEAR INDEPENDENCE

For the rest of the manuscript we restrict attention to nilpotent Springer varieties, i.e., the case in which the Hessenberg function is the identity function $h(i) = i$ for all $1 \le i \le n$. In this section we recount for the convenience of the reader several ideas developed in [2, 9] which are used in the next two sections in the study of a special class of nilpotent Springer varieties. First we recall the **dimension pair algorithm** introduced in [2] which associates to each S^1-fixed point in a nilpotent Hessenberg variety a permutation in S_n. We also recall the interpretation of the algorithm in terms of the poset pinball game introduced in [9]. More specifically, in the case of nilpotent Springer varieties, the algorithm has an interpretation as producing the output of a **successful game of Betti poset pinball**, as is shown in [2, Proposition 3.6]. We keep exposition brief and refer the reader to [2] for details.

We begin with the definition of dimension pairs for the special case of the identity Hessenberg function $h(i) = i$.

Definition 5.1. Let λ a Young diagram with n boxes and T a permissible filling of λ. The pair (a,b) is a **dimension pair** of T if the following conditions hold:

(1) $b > a$,
(2) b is either
   - below a and in the same column, or
   - anywhere in a column strictly to the left of the column of a,
   and
(3) if there exists a box with filling c directly adjacent to the right of a, then $b \le c$.

For a dimension pair (a,b) of T, we will refer to b as the **top part** of the dimension pair.

Example 5.2. *Let $\lambda = (2,2)$. For the permissible filling*

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

the dimension pairs are $\{(1,2),(3,4)\}$.

Given a permissible filling T of λ, denote by DP^T the set of dimension pairs of T. For each integer ℓ with $2 \leq \ell \leq n$, define

(5.1) $$x_\ell := |\{(a, \ell) \mid (a, \ell) \in DP^T\}|.$$

We call the integral vector $\mathbf{x} = (x_2, x_3, \ldots, x_n)$ the **list of top parts** of T. To each such $\mathbf{x}$ we associate a permutation in S_n as follows. As a preliminary step, for each ℓ with $2 \leq \ell \leq n$ define

$$u_\ell(\mathbf{x}) := \begin{cases} s_{\ell-1} s_{\ell-2} \cdots s_{\ell-x_\ell} & \text{if } x_\ell > 0 \\ 1 & \text{if } x_\ell = 0 \end{cases}$$

where s_i denotes the simple transposition $(i, i+1)$ in S_n and 1 denotes the identity permutation. We define an association $\mathbf{x} \mapsto \omega(\mathbf{x}) \in S_n$ by

(5.2) $$\omega(\mathbf{x}) := u_2(\mathbf{x}) u_3(\mathbf{x}) \cdots u_n(\mathbf{x}) \in S_n.$$

With the terminology in place we now recall the **dimension pair algorithm** introduced in [2]. Suppose N is a nilpotent $n \times n$ matrix in Jordan canonical form and weakly decreasing sizes of Jordan blocks, with corresponding Young diagram λ. Following notation in Section 4 denote by T_{RE} the rotated English filling of λ and let $\sigma := \phi_\lambda(T_{RE})$ be the permutation such that $N_{hf} := \sigma N \sigma^{-1}$ is the choice of highest form of N used in [17, Section 4].

Definition of $roll : \mathcal{S}_{N_{hf}}^{S^1} \to S_n$:

(1) Let $w \in \mathrm{Hess}(N_{hf}, h)^{S^1}$ and let $\phi_{\lambda,\sigma}^{-1}(w^{-1})$ be its corresponding permissible filling.
(2) Let $DP^{\phi_{\lambda,\sigma}^{-1}(w^{-1})}$ be the set of dimension pairs in the permissible filling $\phi_{\lambda,\sigma}^{-1}(w^{-1})$.
(3) For each ℓ with $2 \leq \ell \leq n$, set

$$x_\ell := |\{(a, \ell) \mid (a, \ell) \in DP^{\phi_{\lambda,\sigma}^{-1}(w^{-1})}\}|$$

as in (5.1) and define $\mathbf{x} := (x_2, \ldots, x_n)$.
(4) Define $roll(w) := (\omega(\mathbf{x}))^{-1}$ where $\omega(\mathbf{x})$ is the permutation associated to the integer vector $\mathbf{x}$ defined in (5.2).

We call $roll(w)$ the **rolldown** of w, following terminology introduced in [9]. The idea motivating the dimension pair algorithm is that we can interpret the association $w \mapsto roll(w)$ as a result of a game of **poset pinball**, defined in [9, Section 3]. A poset pinball game starts with the data of an ambient partially ordered set, a rank function ℓ on the poset, and a designated subset of the poset (called the **initial subset**); in our setting these are the permutation group S_n equipped with Bruhat order, rank function $\ell : S_n \to \mathbb{Z}$ given by Bruhat length, and the Springer fixed points $\mathcal{S}_{N_{hf}}^{S^1}$ respectively. The **Betti pinball** version of the game then proceeds by assigning to each element of $\mathcal{S}_{N_{hf}}^{S^1}$ a permutation in S_n satisfying certain conditions (see [9, Section 3] for details), one of which concerns the Betti numbers of $\mathcal{S}_{N_{hf}}$. We recall the following result of Tymoczko (reformulated in our language). Although in [17] Tymoczko deals with a more general situation we state her result only for the special case of Springer varieties. The statement assumes that N_{hf} is in the highest form corresponding to the rotated English filling of Definition 3.7.

Theorem 5.3. *([17, Theorem 1.1]) Let $N_{hf} : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent matrix in highest form chosen as above and let $\lambda := \lambda_{N_{hf}}$. Let $\mathcal{S}_{N_{hf}}$ denote the corresponding nilpotent Springer variety. There is a paving by (complex) affine cells of $\mathcal{S}_{N_{hf}}$ such that:*

- *the affine cells are in one-to-one correspondence with $\mathcal{S}_{N_{hf}}^{S^1}$, and*
- *the (complex) dimension of the affine cell C_w corresponding to a fixed point $w \in \mathcal{S}_{N_{hf}}^{S^1}$ is*

(5.3) $$\dim_{\mathbb{C}}(C_w) = |DP^{\phi_{\lambda,\sigma}^{-1}(w^{-1})}|$$

where $\sigma = \phi_\lambda(T_{RE})$.

In particular, Theorem 5.3 implies that the odd Betti numbers of $\mathcal{S}_{N_{hf}}$ are 0, and the $2k$-th even Betti number is precisely the number of fixed points w such that $|DP^{\phi_\lambda^{-1}(w^{-1})}| = k$. In this sense the dimension pairs in the permissible fillings $\phi_{\lambda,\sigma}^{-1}(w^{-1})$ contain the data of the Betti numbers of $\mathcal{S}_{N_{hf}}$. One of the rules of

the Betti pinball game (see [9, Section 3] and [2, Section 3] for more details) is that for every $k \geq 0, k \in \mathbb{Z}$, we must have

$$b_k = \left| \left\{ roll(w) \mid w \in \mathrm{Hess}(N_{hf}, h)^{S^1} \text{ with } \ell(roll(w)) = k \right\} \right|$$

where b_k denotes the $2k$-th Betti number of $\mathcal{S}_{N_{hf}}$. By the definition of $roll : \mathcal{S}_{N_{hf}}^{S^1} \to S_n$ this condition is satisfied. It is shown in [2, Proposition 3.8] that the association $w \mapsto roll(w)$ also satisfies the other necessary conditions to be interpreted in this context as an outcome of a successful game of Betti poset pinball.

For any $u \in S_n$, define the class $p_u := \pi(\sigma_u)$ to be the image of the classical equivariant Schubert class $\sigma_u \in H_T^*(\mathcal{F}lags(\mathbb{C}^n))$ under the projection

$$(5.4) \qquad\qquad \pi : H_T^*(\mathcal{F}lags(\mathbb{C}^n)) \to H_{S^1}^*(\mathcal{S}_{N_{hf}})$$

induced by the inclusion of groups $S^1 \hookrightarrow T$ and the S^1-equivariant inclusion of spaces $\mathcal{S}_{N_{hf}} \hookrightarrow \mathcal{F}lags(\mathbb{C}^n)$. In analogy with the terminology in [2, 10], we refer to the images p_u as **Springer Schubert classes**.

One of the goals of poset pinball is to build explicit module bases for equivariant cohomology rings. In the context of nilpotent Hessenberg varieties, one method by which to do so is to find an appropriate subset of the Hessenberg Schubert classes p_u (defined analogously to the Springer Schubert classes above) which form a module basis for $H_{S^1}^*(\mathrm{Hess}(N_{hf}, h))$. To show that a subset is a basis, we must in particular show that the subset is linearly independent. Using the fact that equivariant Schubert classes satisfy

$$(5.5) \qquad\qquad \sigma_v(w) = 0 \text{ if } w \not\geq v$$

for all $w, v \in S_n$, it follows that if the rolldowns $roll(w)$ of the Hessenberg fixed points satisfy the **poset-upper-triangularity condition**

$$(5.6) \qquad\qquad roll(w) \leq u \Leftrightarrow w \leq u$$

for all $w, u \in \mathcal{S}_{N_{hf}}^{S^1}$, then the corresponding Hessenberg Schubert classes are linearly independent [9, Section 2]. In [2] the results of the dimension pair algorithm is studied in detail for a special case of regular nilpotent Hessenberg varieties $\mathrm{Hess}(N, h)$. In this case it turns out that the set of permutations $\{roll(w)\}_{w \in \mathrm{Hess}(N,h)^{S^1}}$ satisfy the poset-upper-triangularity property (5.6) (see [2, Theorem 4.1]). Combining this poset-upper-triangularity with the fact that the rolldowns obtained by the dimension pair algorithm are compatible with the Betti numbers of $\mathrm{Hess}(N_{hf}, h)$ [2, Lemma 3.6], it then follows from [9, Proposition 4.14] that the corresponding Hessenberg Schubert classes $\{p_{roll(w)}\}_{w \in \mathrm{Hess}(N,h)^{S^1}}$ form a $H_{S^1}^*(\mathrm{pt})$-module basis for $H_{S^1}^*(\mathrm{Hess}(N, h))$ [2, Proposition 3.9].

However, it turns out that in the $(n-2, 2)$ Springer variety case studied in detail below, the rolldowns $\{roll(w)\}_{w \in \mathcal{S}_{N_{hf}}^{S^1}}$ coming from the dimension pair algorithm are not necessarily poset-upper-triangular, as we show below, so we cannot apply [9, Proposition 4.14]. Instead it requires further analysis to determine that the classes $\{p_{roll(w)}\}_{w \in \mathcal{S}_{N_{hf}}^{S^1}}$ are linearly independent; this is the content of Section 7 below. Once linear independence is established we use the following proposition to conclude that the set $\{p_{roll(w)}\}_{w \in \mathcal{S}_{N_{hf}}^{S^1}}$ is a module basis.

Proposition 5.4. *Let* $N : \mathbb{C}^n \to \mathbb{C}^n$ *be a nilpotent operator in standard Jordan canonical form with weakly decreasing Jordan block sizes with corresponding Young diagram* λ. *Let* $\mathcal{S}_{N_{hf}}$ *be the Springer variety corresponding to the highest form* $N_{hf} := \sigma N \sigma^{-1}$ *where* σ *is the permutation corresponding to the rotated English filling of* λ, *equipped with the* S^1 *action defined in* (3.4). *Let* $roll : \mathcal{S}_{N_{hf}}^{S^1} \to S_n$ *be the dimension-pair algorithm defined above. Suppose the classes* $\{p_{roll(w)} \mid w \in \mathcal{S}_{N_{hf}}^{S^1}\}$ *are linearly independent in* $H_{S^1}^*(\mathcal{S}_{N_{hf}})$. *Then the set* $\{p_{roll(w)} \mid w \in \mathcal{S}_{N_{hf}}^{S^1}\}$ *of Springer Schubert classes form a* $H_{S^1}^*(\mathrm{pt})$-*module basis for the* S^1-*equivariant cohomology ring* $H_{S^1}^*(\mathcal{S}_{N_{hf}})$.

Proof. Since $roll : \mathcal{S}_{N_{hf}}^{S^1} \to S_n$ represents a possible outcome of a successful game of Betti poset pinball by [2, Proposition 3.7] the assertion follows from [9, Proposition 4.13]. $\square$

Remark 5.5. *The* $(n-2, 2)$ *Springer variety example studied here is the first example in the poset pinball literature of an instance of successful Betti pinball which does not yield a poset-upper-triangular basis.*

Finally we briefly recall the injectivity results in equivariant cohomology which computationally simplify the proof that the Springer Schubert classes are linearly independent. The next proposition follows

from known results about the topology of Springer varieties [16] and a standard argument in equivariant cohomology (see e.g. [9, Remark 4.11 and Proposition 6.2]).

Proposition 5.6. *Let $N : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent operator in standard Jordan canonical form with weakly decreasing Jordan block sizes with corresponding Young diagram λ. Let $\mathcal{S}_{N_{hf}}$ be the Springer variety corresponding to the highest form $N_{hf} := \sigma N \sigma^{-1}$ where σ is the permutation corresponding to the rotated English filling of λ, equipped with the S^1-action defined in (3.4). Then the inclusion $\iota : \mathcal{S}_{N_{hf}}^{S^1} \hookrightarrow \mathcal{S}_{N_{hf}}$ induces an injection in S^1-equivariant cohomology*

$$\iota^* : H_{S^1}^*(\mathcal{S}_{N_{hf}}) \hookrightarrow H_{S^1}^*(\mathcal{S}_{N_{hf}}^{S^1}) \cong \bigoplus_{w \in \mathcal{S}_{N_{hf}}^{S^1}} H_{S^1}^*(\mathrm{pt}) \cong \bigoplus_{w \in \mathcal{S}_{N_{hf}}^{S^1}} \mathbb{C}[t].$$

The above proposition implies that a Springer Schubert class p_u in $H_{S^1}^*(\mathcal{S}_{N_{hf}})$ can be specified by $\iota^*(p_u)$, which we view as a vector of polynomials in $\mathbb{C}[t]$ with coordinates indexed by the fixed points $\mathcal{S}_{N_{hf}}^{S^1}$. Following notation of [2, 9, 10], we denote by $p_u(w)$ the w-th coordinate of $\iota^*(p_u)$. The next result, which we use later, is straightforward.

Proposition 5.7. *Let $N, \lambda, \mathcal{S}_{N_{hf}}$ be as above. If the columns of the matrix*

$$(p_{roll(w)}(u))_{w,u \in \mathcal{S}_{N_{hf}}^{S^1}}$$

(where the variable w is the index of the columns and u the index of the rows) are linearly independent over $H_{S^1}^*(\mathrm{pt}) \cong \mathbb{C}[t]$, then the set of Springer Schubert classes $\{p_{roll(w)}\}_{w \in \mathcal{S}_{N_{hf}}^{S^1}}$ is linearly independent.

6. SMALL-n CASES: $n = 4$ AND $n = 5$

In this section and Section 7 we restrict attention to the nilpotent Springer varieties corresponding to Young diagrams of the form $(n-2,2)$ for $n \geq 4$. In this setting we denote by $\mathcal{S}_{(n-2,2)}$ the Springer variety $\mathcal{S}_{N_{hf}}$ corresponding to the nilpotent matrix $N_{hf} := \sigma N \sigma^{-1}$ in highest form with associated Young diagram $(n-2,2)$ where σ is the permutation corresponding to the rotated English filling of $(n-2,2)$. The goal, as explained in Section 5, is to prove that the dimension pair algorithm produces in this case a module basis for $H_{S^1}^*(\mathcal{S}_{(n-2,2)})$. To this end we concretely compute the Springer fixed points, associated permissible fillings, dimension pairs, and rolldowns for the cases $n = 4$ and $n = 5$, i.e. for the Springer varieties corresponding to the Young diagrams

 and
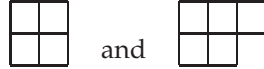
We also explicitly check in these cases that the corresponding Springer Schubert classes are poset-upper-triangular and hence linearly independent. The inductive argument we give in the next section requires the $n = 4$ case as its base case. We choose to additionally explicitly compute and record the $n = 5$ case because it suggests the outline of the general inductive argument.

Below we present two tables of data. The columns correspond to the following:

- w: an S^1-fixed point in the Springer variety $\mathcal{S}_{(n-2,2)}$.
- w^{-1}: the inverse of w.
- *perm filling*: the permissible filling $\phi_{\lambda,\sigma}^{-1}(w^{-1})$.
- *dim pair*: the dimension pairs of the permissible filling.
- *deg*: the number of dimension pairs of the permissible filling (equivalently, the cohomology degree of the associated Springer Schubert class).
- $\omega(\mathbf{x})$ the permutation associated to the list $\mathbf{x}$ of "top parts" of the dimension pairs.
- $roll(w)$: inverse of $\omega(\mathbf{x})$, and by definition of the dimension pair algorithm, the rolldown of w.

Example 6.1. *Let $n = 4$ and $\lambda_N = (2,2)$. The following table records the data outlined above. Part of these computations are also contained in [13].*

From this table it can be seen explicitly that the only Springer fixed point w in $\mathcal{S}_{(2,2)}$ with $roll(w) \neq w$ is $w = 2413$. Moreover it is straightforward to check that the rolldown 1423 of $w = 2413$ is not Bruhat-less than any of the other Springer fixed points. These facts together imply that these Springer fixed points and associated rolldowns satisfy the poset-upper-triangularity property

(6.1) $$roll(w) \leq u \Leftrightarrow w \leq u$$

TABLE 1. Dimension pair data for the Springer variety $\mathcal{S}_{(2,2)}$.

w	w^{-1}	perm filling	dim pair	deg	$\omega(\mathbf{x})$	$roll(w)$
$1234 = e$	1234	$\begin{array}{\|c\|c\|}\hline 2 & 4 \\\hline 1 & 3 \\\hline\end{array}$	$\emptyset$	0	1234	$1234 = e$
$2134 = s_1$	2134	$\begin{array}{\|c\|c\|}\hline 1 & 4 \\\hline 2 & 3 \\\hline\end{array}$	$\{(1,2)\}$	1	2134	$2134 = s_1$
$1324 = s_2$	1324	$\begin{array}{\|c\|c\|}\hline 3 & 4 \\\hline 1 & 2 \\\hline\end{array}$	$\{(2,3)\}$	1	1324	$1324 = s_2$
$1243 = s_3$	1243	$\begin{array}{\|c\|c\|}\hline 2 & 3 \\\hline 1 & 4 \\\hline\end{array}$	$\{(3,4)\}$	1	1243	$1243 = s_3$
$2143 = s_1s_3$	2143	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\\hline 2 & 4 \\\hline\end{array}$	$\{(1,2),(3,4)\}$	2	2143	$2143 = s_1s_3$
$2413 = s_1s_3s_2$	3142	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\\hline 3 & 4 \\\hline\end{array}$	$\{(2,4),(2,3)\}$	2	1342	$1423 = s_3s_2$

for all fixed points w, u. By an argument identical to [2, Lemma 4.4] which uses the poset-upper-triangularity property (5.5) of the equivariant Schubert classes $\{\sigma_w\}_{w \in S_n}$, this implies that the Springer Schubert classes $\{p_{roll(w)}\}_{w \in \mathcal{S}^{S^1}_{(2,2)}}$ are poset-upper-triangular and hence linearly independent and a $H^*_{S^1}(\mathrm{pt})$-module basis for $H^*_{S^1}(\mathcal{S}_{(2,2)})$.

We have just explicitly checked that in the case $n = 4$, the dimension pair algorithm interpreted in terms of Betti pinball produces a module basis of $H^*_{S^1}(\mathcal{S}_{(2,2)})$. We now compute the $n = 5$ case and relate it to the $n = 4$ case, thereby illustrating the outline of the general inductive argument.

Example 6.2. *Let $n = 5$ and $\lambda = (3, 2)$. Suppose T is a permissible filling of $(3, 2)$ where the entry 5 is in the top row. Since the rows in a permissible filling are increasing this means that the 5 occurs in the rightmost box of the top row of T. Deleting this box yields a valid permissible filling of $(2, 2)$ which therefore occurs in the previous $n = 4$ example. For permissible fillings T of this form the corresponding fixed point w and its rolldown are easily seen to be identical to those obtained in the previous example (viewed as elements of S_5 instead of S_4 via the usual embedding $S_4 \hookrightarrow S_5$). Hence the permissible fillings in the $n = 5$ case which do not occur in the $n = 4$ case are precisely those for which the entry 5 is in the bottom row. There are four such permissible fillings as may be seen in the table below.*

We claim that, as in the $n = 4$ case, the rolldowns satisfy the condition (6.1), which then implies by the same argument that the corresponding Springer Schubert classes are poset-upper-triangular and hence linearly independent and a module basis. To prove this claim it suffices to check (6.1) for those w for which $roll(w) \neq w$. We check each case by hand.

For $w = s_1s_3s_2$ with $roll(w) = s_3s_2$, we see that $roll(w) < s_3s_4s_1s_2$ and $roll(w) < s_3s_4s_1s_2s_3$. Since also $w < s_3s_4s_1s_2$ and $w < s_3s_4s_1s_2s_3$, the claim holds in this case. Next observe that the last four fixed points in the above table are linearly ordered with respect to the Bruhat order, i.e.

$$s_3s_4 < s_3s_4s_1 < s_3s_4s_1s_2 < s_3s_4s_1s_2s_3.$$

In the case of $w = s_3s_4$ we have $roll(w) = s_4$. Moreover s_4 is not Bruhat-less than any of the fixed points occurring in the $n = 4$ case and is Bruhat-less than all of the last four fixed points, so the claim holds in this case. Similarly, the rolldowns for the last three fixed points satisfy

$$roll(s_3s_4s_1) = s_4s_1 \not< s_3s_4, \quad roll(s_3s_4s_1s_2) = s_4s_2 \not< s_3s_4s_1, \quad roll(s_3s_4s_1s_2s_3) = s_4s_3 \not< s_3s_4s_1s_2,$$

so the claim holds in all cases. This proves the claim and hence that the Springer Schubert classes are poset-upper-triangular in the $n = 5$ case and hence a module basis, as desired.

7. A POSET PINBALL MODULE BASIS FOR $(n-2, 2)$ SPRINGER VARIETIES

The main result of this section is that the dimension pair algorithm produces a set of Springer Schubert classes $\{p_{roll(w)}\}_{w \in \mathcal{S}^{S^1}_{(n-2,2)}}$ which are a module basis, in the case of $(n-2, 2)$ Springer varieties for any $n \geq 4$. We have the following.

TABLE 2. Dimension pair data for the Springer variety $\mathcal{S}_{(3,2)}$.

w	w^{-1}	perm filling	dim pair	deg	$\omega(\mathbf{x})$	$roll(w)$
$12345 = e$	12345	$\begin{array}{ccc}2&4&5\\1&3\end{array}$	$\emptyset$	0	12345	$12345 = e$
$21345 = s_1$	21345	$\begin{array}{ccc}1&4&5\\2&3\end{array}$	$\{(1,2)\}$	1	21345	$21345 = s_1$
$13245 = s_2$	13245	$\begin{array}{ccc}3&4&5\\1&2\end{array}$	$\{(2,3)\}$	1	13245	$13245 = s_2$
$12435 = s_3$	12435	$\begin{array}{ccc}2&3&5\\1&4\end{array}$	$\{(3,4)\}$	1	12435	$12435 = s_3$
$21435 = s_1s_3$	21435	$\begin{array}{ccc}1&3&5\\2&4\end{array}$	$\{(1,2),(3,4)\}$	2	21435	$21435 = s_1s_3$
$24135 = s_1s_3s_2$	31425	$\begin{array}{ccc}1&2&5\\3&4\end{array}$	$\{(2,3),(2,4)\}$	2	13425	$14235 = s_3s_2$
$12453 = s_3s_4$	12534	$\begin{array}{ccc}2&3&4\\1&5\end{array}$	$\{(4,5)\}$	1	12354	$12354 = s_4$
$21453 = s_3s_4s_1$	21534	$\begin{array}{ccc}1&3&4\\2&5\end{array}$	$\{(1,2),(4,5)\}$	2	21354	$21354 = s_4s_1$
$24153 = s_3s_4s_1s_2$	31524	$\begin{array}{ccc}1&2&4\\3&5\end{array}$	$\{(2,3),(4,5)\}$	2	13254	$13254 = s_4s_2$
$24513 = s_3s_4s_1s_2s_3$	41523	$\begin{array}{ccc}1&2&3\\4&5\end{array}$	$\{(3,4),(3,5)\}$	2	12453	$12534 = s_4s_3$

Theorem 7.1. *Let $n \geq 4$. Let $N : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent operator in standard Jordan canonical form with weakly decreasing Jordan block sizes $n - 2$ and 2. Let $N_{hf} := \sigma N \sigma^{-1}$ be the choice of highest form of N where σ is the permutation corresponding to the rotated English filling of $(n-2,2)$. Let $\mathcal{S}_{(n-2,2)}$ be the Springer variety corresponding to N_{hf} equipped with the S^1-action defined in (3.4). Let $roll : \mathcal{S}_{(n-2,2)}^{S^1} \to S_n$ be the function defined by the dimension-pair algorithm. Then the columns of the matrix*

$$(p_{roll(w)}(u))_{w,u \in \mathcal{S}_{(n-2,2)}^{S^1}}$$

with entries in $H_{S^1}^*(\mathrm{pt}) \cong \mathbb{C}[t]$ are linearly independent over $H_{S^1}^*(\mathrm{pt})$. (Here w is the variable indexing the columns and u the index of the rows.) In particular, the Springer Schubert classes $\{p_{roll(w)}\}_{w \in \mathcal{S}_{(n-2,2)}^{S^1}}$ form a $H_{S^1}^*(\mathrm{pt})$-module basis for the equivariant cohomology ring $H_{S^1}^*(\mathcal{S}_{(n-2,2)})$ of the Springer variety.

Remark 7.2. *The above theorem extends the subregular Springer case (which corresponds to Young diagrams of shape $(n-1,1)$), for which it was shown in [9] that the set of Springer Schubert classes obtained by the dimension pair algorithm is poset-upper-triangular, so in particular linearly independent. (Although the results in [9] are not phrased using the terminology of this paper it is straightforward to see that the classes used in [9] agree with those arising from the dimension pair algorithm.)*

Since the rows are increasing in a Springer permissible filling, we can naturally decompose the set of $(n-2,2)$ permissible fillings into two subsets: namely, those for which the largest entry n occupies the top row, and those for which n occupies the bottom row. As observed in Example 6.2 above, when n is in the top row, the permissible filling obtained by removing the rightmost box in the top row is a permissible filling for the Young diagram $(n-3,2)$, corresponding to the smaller Springer variety $\mathcal{S}_{(n-3,2)}$. This sets us up for an inductive argument. Since we have already seen in Section 6 the linear independence for the cases $n = 4$ and $n = 5$, we start the induction start at $n = 6$. We begin with a preliminary lemma generalizing the observations made in Example 6.2.

Lemma 7.3. *Let $n \geq 6$. Let $N, N_{hf}, \mathcal{S}_{(n-2,2)}$ and $roll$ be as in Theorem 7.1. Then*

- *there are precisely $n - 1$ permissible fillings of $(n-2,2)$ with n in the bottom row,*
- *the $n - 1$ such permissible fillings, their corresponding Springer fixed points w, and their rolldowns $roll(w)$ are precisely those listed in the table below,*

- *these $n-1$ Springer fixed points are linearly ordered with respect to Bruhat order, i.e.*

(7.1) $s_3 s_4 \cdots s_{n-2} s_{n-1} < s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 < \cdots < s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 \cdots s_{n-3} s_{n-2}.$

TABLE 3. Dimension pair algorithm data for the Springer fixed points in $\mathcal{S}^{S^1}_{(n-2,2)}$ corresponding to permissible fillings with n in the bottom row.

pf	w^{-1}	w	v_{hf}
$\begin{array}{ccccc}2&3&4&\cdots&n-1\\1&n\end{array}$	$1\,2\,n\,3\,4\cdots$	$s_3 s_4 \cdots s_{n-2} s_{n-1}$	s_{n-1}
$\begin{array}{ccccc}1&3&4&\cdots&n-1\\2&n\end{array}$	$2\,1\,n\,3\,4\cdots$	$s_3 s_4 \cdots s_{n-2} s_{n-1} s_1$	$s_{n-1} s_1$
$\begin{array}{ccccc}1&2&4&\cdots&n-1\\3&n\end{array}$	$3\,1\,n\,2\,4\cdots$	$s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2$	$s_{n-1} s_2$
$\begin{array}{ccccc}1&2&3&\cdots&n-1\\4&n\end{array}$	$4\,1\,n\,2\,3\cdots$	$s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 s_3$	$s_{n-1} s_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\begin{array}{cccc}1&2&\cdots&n-3\;n-2\\n-1&n\end{array}$	$n-1\,1\,n\,2\,3\cdots$	$s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 \cdots s_{n-3} s_{n-2}$	$s_{n-1} s_{n-2}$

Proof of Lemma 7.3. Since the Springer permissible fillings are precisely those which are row-strict, it is immediate that the permissible fillings listed in the table are precisely those with n in the bottom row. In particular there are exactly $n-1$ such permissible fillings as claimed. Moreover, it follows from the definition of $\phi_{\lambda,\sigma}$ (which corresponds to the rotated English reading) that the one-line notation of the w^{-1} are those given in the table. Explicit computation also verifies that the following expressions in the simple transpositions are indeed reduced word decompositions of the w^{-1}:

- $1\,2\,n\,3\,4\cdots = s_{n-1} s_{n-2} \cdots s_4 s_3$
- $2\,1\,n\,3\,4\cdots = s_1 s_{n-1} s_{n-2} \cdots s_4 s_3$
- $3\,1\,n\,2\,4\cdots = s_2 s_1 s_{n-1} s_{n-2} \cdots s_4 s_3$
- $4\,1\,n\,2\,3\cdots = s_3 s_2 s_1 s_{n-1} s_{n-2} \cdots s_4 s_3$
- $\vdots$
- $n-1\,1\,n\,2\,3\cdots = s_{n-2} s_{n-1} \cdots s_2 s_1 s_{n-1} s_{n-2} \cdots s_4 s_3,$

from which it follows that the w are those given in the list. For k with $1 \le k \le n-1$, the definition of dimension pairs implies that the permissible filling with k and n in the bottom row contains as dimension pairs $\{(1,k),(n-1,n)\}$ for $2 \le k \le n-1$ and $\{(n-1,n)\}$ for $k=1$. From this it follows from the definition of $\omega(\mathbf{x})$ that $roll(w)$ is as given in the table. Finally, from the given reduced word decompositions and the definition of Bruhat order we obtain (7.1) as desired.   □

Before proceeding with the proof of Theorem 7.1 we briefly recall the **Billey formula** for computing restrictions $\sigma_v(w)$ of Schubert classes σ_v at some w in S_n. We use the formulation given in [11]. Let α_i denote the simple root $t_i - t_{i+1}$ and $\widehat{\alpha}_i$ the operator on $H^*_T(\mathrm{pt})$ which multiplies by α_i.

Theorem 7.4. *([3, Theorem 4], also cf. [11]) Suppose I is a reduced word expression for $w \in S_n$. For each $v \in S_n$ we have*

(7.2) $\sigma_v(w) = \sum_{J \subseteq I} \prod_{i \in I} \left(\widehat{\alpha}_i^{[i \in J]} r_i \right) \cdot 1$

where the sum is over reduced subwords J of I with product v, the notation $\widehat{\alpha}_i^{[i \in J]}$ means that $\widehat{\alpha}_i$ is included only if $i \in J$, and r_i is the reflection corresponding to s_i.

We record the following fact, used in the proof below, which follows straightforwardly from the Billey formula.

Fact 7.5. *Suppose $v, w \in S_n$ with $v \leq w$ in Bruhat order. Suppose there exists a decomposition $w = w' \cdot w''$ for $w', w'' \in S_n$ where $v \leq w'$ and, for all simple transpositions s_i such that $s_i < v$, we have $s_i \not\leq w''$. Then $\sigma_v(w) = \sigma_v(w')$.*

As explained in Section 5, we need to compute the restrictions $p_{roll(w)}(u)$ for w, u Springer fixed points. Since $p_{roll(w)}$ is by definition the image of the equivariant Schubert class $\sigma_{roll(w)}$ under the ring map (5.4) and because the diagram

$$
\begin{array}{ccc}
H_T^*(\mathcal{F}lags(\mathbb{C}^n)) \hookrightarrow & H_T^*((\mathcal{F}lags(\mathbb{C}^n))^T) \cong \bigoplus_{w \in W} H_T^*(\mathrm{pt}) \\
\downarrow & & \downarrow \\
H_{S^1}^*(\mathcal{S}_{(n-2,2)}) \hookrightarrow & H_{S^1}^*(\mathcal{S}_{(n-2,2)})^{S^1}) \cong \bigoplus_{w \in \mathrm{Hess}(h)^{S^1}} H_{S^1}^*(\mathrm{pt})
\end{array}
$$

commutes, the polynomial $p_{roll(w)}(u) \in H_{S^1}^*(\mathrm{pt}) \cong \mathbb{C}[t]$ can be computed by first evaluating $\sigma_{roll(w)}(u)$ by the Billey formula (7.2) and then using the linear projection $\mathfrak{t}^* \to \mathrm{Lie}(S^1)^*$ for our choice of S^1 in (3.4) given in Lemma 3.28. We use this technique repeatedly in the proof below.

Proof of Theorem 7.1 . By Propositions 5.6 and 5.7, it suffices to prove that the matrix obtained from the restrictions to fixed points

$$
(p_{roll(w)}(u))_{w,u \in \mathcal{S}_{(n-2,2)}^{S^1}}
$$

has $H_{S^1}^*(\mathrm{pt})$-linearly independent columns.

Let $n \geq 4$. We have seen in Section 6 that the above assertion holds for the cases $n = 4$ and $n = 5$. Hence assume now that $n \geq 6$. We assume by induction that for the $n-1$ case, i.e. for the case of the partition $(n-3, 2)$, the above matrix has linearly independent columns.

For concreteness and for the remainder of the argument, we assume that the fixed points $w \in \mathcal{S}_{(n-2,2)}^{S^1}$ have been linearly ordered so that the fixed points corresponding to permissible fillings containing the n in the top row appear first, and that the fixed points associated to fillings with n in the bottom row are given the ordering in the table in Lemma 7.3 (reading from top to bottom). Ordered in this manner, we may write the above matrix in terms of submatrices as follows:

$$
(7.3) \qquad (p_{roll(w)}(u))_{w,u \in \mathcal{S}_{(n-2,2)}^{S^1}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
$$

where the submatrix A has entries $p_{roll(w)}(u)$ where both w, u correspond to fillings with n in the top row, D corresponds to those where both w, u have n in the bottom row, and so on.

Consider the submatrix A. For an entry $p_{roll(w)}(u)$ in A, by assumption w is in the subgroup $S_{n-1} \subseteq S_n$ and it is straightforward to see from the definition of the dimension pair algorithm that $roll(w)$ is equal to the rolldown of w considered as an element of $\mathcal{S}_{(n-3,2)}^{S^1}$. Since $u \in S_{n-1}$ also this submatrix is equal to the matrix of restrictions to fixed points obtained in the $(n-3, 2)$ case and so by induction A has linearly independent columns.

Next consider the submatrix B corresponding to $p_{roll(w)}(u)$ where $\phi_{\lambda,\sigma}^{-1}(w^{-1})$ has n in the bottom row and $\phi_{\lambda,\sigma}^{-1}(u^{-1})$ has n in the top row. From Lemma 7.3 and the table given there, we know that the rolldown $roll(w)$ of any such w contains the simple transposition s_{n-1} in its reduced word decomposition. On the other hand, for u with n in the top row, u is an element in the subgroup S_{n-1} which fixes the element n, and in particular a reduced word decomposition for u may be written solely with the simple transpositions $s_1, s_2, \ldots, s_{n-2}$. Hence $roll(w) \not\leq u$ in Bruhat order, and by the upper-triangularity property (5.5) of equivariant Schubert classes this implies $p_{roll(w)}(u) = 0$. We conclude that the entire submatrix is 0 and the matrix (7.3) is in fact of the form

$$
\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}
$$

where A has linearly independent columns. In order to prove that the full matrix has linearly independent columns, we wish to prove that the submatrix D has linearly independent columns. The remainder of the proof is dedicated to the justification of this last claim, for which we explicitly compute the appropriate entries $p_{roll(w)}(u)$ using the Billey formula (7.2).

We compute each column of D in the linear order given by the enumeration in Lemma 7.3 of those w with n in the bottom row. For the Billey computations below we use the choices of reduced word decompositions for w and $roll(w)$ given in the same lemma.

First consider the case $w = s_3 s_4 \cdots s_{n-2} s_{n-1}$. Then $roll(w) = s_{n-1}$. We claim $\sigma_{s_{n-1}}$ evaluates to $t_3 - t_n$ at all fixed points u. Indeed, recalling that the reflection r_i acts on the variables t_j by $r_i(t_i) = t_{i+1}, r_i(t_{i+1}) = t_i$ and $r_i(t_j) = t_j$ for all $j \neq i, i+1$, we have for instance

$$\sigma_{s_{n-1}}(s_3 s_4 \cdots s_{n-2} s_{n-1}) = r_3 r_4 \cdots r_{n-2}(t_{n-1} - t_n)$$
$$= r_3 r_4 \cdots r_{n-3}(t_{n-2} - t_n)$$

(7.4)
$$\vdots$$
$$= t_3 - t_n,$$

which proves the claim for $u = w$. For all other u with n in the bottom row, the computation of the Billey formula differs from (7.4) only in that there are extra simple transpositions occurring after the s_{n-1} in the reduced word decomposition of u. By Fact 7.5 these extra transpositions make no difference in the evaluation of $\sigma_{s_{n-1}}(u)$ and so $\sigma_{s_{n-1}}(u) = t_3 - t_n$ for all u. The restriction $p_{roll(w)}(u) = p_{s_{n-1}}(u)$ is equal to the image of $\sigma_{s_{n-1}}(u) \in H_T^*(\mathrm{pt})$ under the projection map $H_T^*(\mathrm{pt}) \to H_{S^1}^*(\mathrm{pt})$ induced from the inclusion $S^1 \hookrightarrow T$. By Lemma 3.28 we know $t_3 \mapsto t$ and $t_n \mapsto (n+3-n)t = 3t$ under this projection, from which we conclude that the first (leftmost) column of D is

$$\begin{bmatrix} (t - 3t) = -2t \\ -2t \\ \vdots \\ -2t \end{bmatrix}.$$

Next consider the case $w = s_3 s_4 \cdots s_{n-2} s_{n-1} s_1$ and $roll(w) = s_{n-1} s_1$. In this case, $\sigma_{s_{n-1} s_1}(s_3 s_4 \cdots s_{n-2} s_{n-1}) = 0$ since $s_{n-1} s_1$ does not occur as a subword of $s_3 s_4 \cdots s_{n-2} s_{n-1}$. Also, $\sigma_{s_{n-1} s_1}$ evaluates to $(t_3 - t_n)(t_1 - t_2)$ at all other u. This can be seen from the computation

$$\sigma_{s_{n-1} s_1}(s_3 s_4 \cdots s_{n-2} s_{n-1} s_1) = (r_3 r_4 \cdots r_{n-2}(t_{n-1} - t_n))(r_3 r_4 \cdots r_{n-2} r_{n-1}(t_1 - t_2))$$

(7.5)
$$= (r_3 r_4 \cdots r_{n-2}(t_{n-1} - t_n))(t_1 - t_2)$$
$$= (t_3 - t_n)(t_1 - t_2)$$

for the case $w = s_3 s_4 \cdots s_{n-2} s_{n-1} s_1$. The computation at other u follows from (7.5) and Fact 7.5. Applying Lemma 3.28 again we obtain that the column corresponding to this w is

$$\begin{bmatrix} 0 \\ 2(n-2)t^2 \\ 2(n-2)t^2 \\ \vdots \\ 2(n-2)t^2 \end{bmatrix}.$$

Next consider the case $w = s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2$ and $roll(w) = s_{n-1} s_2$. In this case

$$\sigma_{s_{n-1} s_2}(s_3 s_4 \cdots s_{n-2} s_{n-1}) = \sigma_{s_{n-1} s_2}(s_3 s_4 \cdots s_{n-2} s_{n-1} s_1) = 0$$

since there are no reduced subwords in $s_3 s_4 \cdots s_{n-2} s_{n-1}$ equal to $roll(w) = s_{n-1} s_2$. Furthermore, $\sigma_{s_{n-1} s_2}$ evaluates to $(t_1 - t_4)(t_3 - t_n)$ on all other u. Since the computations are similar to those given above we henceforth keep explanation brief. We have

$$\sigma_{s_{n-1} s_2}(s_3 s_4 \cdots s_{n-1} s_1 s_2) = (t_3 - t_n)(t_1 - t_4)$$

and at other u the computation is similar. Hence the column corresponding to this w is

$$\begin{bmatrix} 0 \\ 0 \\ 2(n-3)t^2 \\ \vdots \\ 2(n-3)t^2 \end{bmatrix}.$$

Next consider the case $w = s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 s_3$ where $roll(w)$ is $s_{n-1} s_3$. In this case, by arguments similar to those above, $\sigma_{s_{n-1} s_3}$ evaluates to $(t_3 - t_4)(t_3 - t_n)$ on the first 3 fixed points listed in the table in Lemma 7.3. Moreover $\sigma_{s_{n-1} s_3}$ evaluates to $(t_3 - t_4)(t_3 - t_n) + (t_1 - t_5)(t_3 - t_n)$ at all other u. We conclude the column corresponding to this w is

$$
\begin{bmatrix}
2nt^2 \\
2nt^2 \\
2nt^2 \\
4(n-1)t^2 \\
\vdots \\
4(n-1)t^2
\end{bmatrix},
$$

where there are $(n-1) - 3 = n - 4$ entries of the form $4(n-1)t^2$.

For the next case, suppose $n \geq 7$. (In the special case $n = 6$, this case is vacuous.) Suppose $k \in \mathbb{Z}$ with $4 \leq k \leq n - 3$. Let $w = s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 \cdots s_{k-1} s_k$ and $roll(w) = s_{n+1} s_k$. By assumption on k, the simple transposition s_k commutes with s_{n-1}. In this case $\sigma_{s_{n-1} s_k}$ evaluates to $(t_3 - t_{k+1})(t_3 - t_n)$ on all fixed points listed in Lemma 7.3 up to $s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 \cdots s_{k-1}$. There are k fixed points in all of this form. Moreover, $\sigma_{s_{n-1} s_k}$ evaluates to $(t_3 - t_{k+1})(t_3 - t_n) + (t_1 - t_{k+2})(t_3 - t_n)$ on the remaining fixed points u which contain $s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 \cdots s_{k-1} s_k$. Hence when projected to $H^*_{S^1}(\mathrm{pt})$, the column corresponding to such a w is

$$
\begin{bmatrix}
2(n-k+3)t^2 \\
\vdots \\
2(n-k+3)t^2 \\
(2(n-k+3) + 2(n-k+1))t^2 \\
\vdots \\
(2(n-k+3) + 2(n-k+1))t^2
\end{bmatrix}
$$

where there are k entries of the form $2(n-k+3)t^2$ and $n-1-k$ entries of the form $(2(n-k+3)+2(n-k+1))t^2$.

Finally, consider the case $w = s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 \ldots s_{n-3} s_{n-2}$ and $roll(w) = s_{n-1} s_{n-2}$. Since s_{n-1} and s_{n-2} do not commute, this computation is somewhat different from the ones given above; in particular $roll(w)$ is *not* Bruhat-less than any of the fixed points u except for the last one listed in Lemma 7.3. Hence in this case $\sigma_{s_{n-1} s_{n-2}}(u) = 0$ at all u except for $u = s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 \cdots s_{n-3} s_{n-2}$, and at this last u, we can compute

$$
\sigma_{s_{n-1} s_n}(s_3 s_4 \cdots s_{n-2} s_{n-1} s_1 s_2 \cdots s_{n-3} s_{n-2}) = (t_3 - t_n)(t_1 - t_n).
$$

Hence the column corresponding to this last w is

$$
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
2t^2
\end{bmatrix}.
$$

We now prove that the columns p_w for w as above are linearly independent over the ring $H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$. The first column p_w with $w = s_3 s_4 \cdots s_{n-2} s_{n-1}$ has a $-2t$ in each entry. We may add or subtract any multiple of this column to or from any other column, and if the resulting set of columns is linearly independent, then so is the original set of columns. It is straightforward to check that for all k with $3 \leq k \leq n - 1$, subtracting $2(n - k + 3)$ times the first column from the column corresponding to w with rolldown $roll(w) = s_{n-1} s_k$ yields

$$
\begin{bmatrix}
0 \\
\vdots \\
0 \\
2(n-k+1)t^2 \\
\vdots \\
2(n-k+1)t^2
\end{bmatrix}
$$

where there are k zeroes at the top of the column and $(n-1)-k$ entries at the bottom of the form $2(n-k+1)t^2$. In particular, adjusted in this manner, the resulting matrix is lower-triangular with non-zero entries along the diagonal, so its columns are linearly independent. As argued above, this implies that the matrix D has linearly independent columns, as was desired. This completes the proof. $\qquad\square$

8. Open questions

We close with some open questions for future work.

Question 8.1. *The computations in the proof of Theorem 7.1 explicitly show that the set of classes* $\{p_{ro\ell\ell(w)}\}_{w\in\mathcal{S}^{S^1}_{(n-2,2)}}$ *are not poset-upper-triangular for $n \geq 6$ since the submatrix D discussed in the proof has non-zero entries both above and below its main diagonal. However the proof also shows that a simple change of basis does yield a poset-upper-triangular basis. We do not know whether this is an instance of a more general phenomenon. It would be of interest to clarify the situation for other cases of Springer varieties.*

Question 8.2. *Both Tymoczko's paving by affines of Hessenberg varieties and the interpretation of our dimension pair algorithm via poset pinball depend on using a Hessenberg variety $\mathrm{Hess}(N, h)$ for which the nilpotent operator N is in highest form. In the case of Tymoczko's paving, this choice can be viewed as a matter only of convenience in the sense that any other translated Hessenberg variety $\mathrm{Hess}(\sigma N\sigma^{-1}, h)$ can be given a paving simply by using translated Schubert cells $\sigma \cdot BwB$ instead of the usual Schubert cells BwB. On the other hand, the poset pinball game delicately depends on the choice of initial subset*

$$\mathrm{Hess}(N, h)^{S^1} \subseteq S_n.$$

Although the sets $\mathrm{Hess}(N, h)^{S^1}$ and $\mathrm{Hess}(\sigma N\sigma^{-1}, h)^{S^1}$ are also related by a simple translation by σ, multiplication by a permutation does not *preserve Bruhat order, so pinball results do not immediately translate from $\mathrm{Hess}(N, h)$ to $\mathrm{Hess}(\sigma N\sigma^{-1}, h)$. One of the main results of this manuscript is that, for a certain special family of Hessenberg varieties $\mathrm{Hess}(N, h) = \mathcal{S}_{N_{hf}}$ (where N_{hf} is a particular choice of highest form) we can use the poset pinball and the dimension pair algorithm to obtain a module basis for $H^*_{S^1}(\mathcal{S}_{N_{hf}})$.*

(1) *It seems plausible that there may be other choices of highest forms (cf. Theorem 3.21), different from that used in this manuscript, which are particularly well-suited for poset pinball.*

(2) *Furthermore, among the choices of highest forms which behave well for poset pinball, there may also be choices best suited for further applications of pinball bases. More specifically, there may be choices highest forms N_T such that a pinball basis for $H^*_{S^1}(\mathcal{S}_{N_T})$ has good properties when mapped to $H^*_{S^1}(\mathcal{S}^{S^1}_{N_T})$. Such choices could then prove useful for e.g. constructions of representations on equivariant cohomology (analogous to the lifts of the classical Springer representations constructed via pinball in [9]).*

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